Lecture 20: Frobenius's Method

Today we continue with using power series to solve linear homogeneous DEs $Lf = 0$, but this time with

$$L = D^2 + \frac{P(x)}{x-x_0} D + \frac{Q(x)}{(x-x_0)^2} \quad \left(\text{and } P, Q \in \mathcal{A}(I),\ I = (x_0-\rho, x_0+\rho)\right).$$

Such an operator is said to have a regular singularity at x_0 (assuming $P(x_0)$, $Q(x_0)$, $Q'(x_0)$ don't all vanish), which means in practical terms that the basis $\{u_1, u_2\}$ of solutions won't both simply be power series — either some non-integer powers of $(x-x_0)$ will be involved or a $\log(x-x_0)$ in one of the $\{u_j\}$.

To simplify, shift coordinates so $x_0 = 0$ and multiply L on the left by x^2 to get

$$L = x^2 D^2 + xP(x) D + Q(x),$$

where $P(x) = \sum_{k=0}^{\infty} p_k x^k$ and $Q(x) = \sum_{k=0}^{\infty} q_k x^k$ on $I = (-\rho, \rho)$.

We seek a solution (on I) of the form

$$(*) \qquad f(x) = \underbrace{|x|^r \sum_{m=0}^{\infty} a_m x^m}_{\text{i.e. } \sum_{m\geq 0} a_m x^{m+r}}, \quad \text{with } a_0 = 1.$$

or just x if $x>0$

Compute $(x>0)$

$$\underline{0} = Lf = \cancel{x^2} \sum_{n\geq 0} (n+r)(n+r-1)\, a_n x^{n+r-\cancel{2}}$$

$$+ \left(\sum_{k\geq 0} p_k x^k\right) \cancel{x} \sum_{m\geq 0} (m+r)\, a_m x^{m+r-\cancel{1}}$$

$$+ \left(\sum_{k\geq 0} q_k x^k\right) \sum_{m\geq 0} a_m x^{m+r}$$

$$= \sum_{n\geq 0} \underbrace{\left\{ (n+r)(n+r-1) a_n + \sum_{m=0}^{n} p_{n-m}(m+r) a_m + \sum_{m=0}^{n} q_{n-m} a_m \right\}}_{\Leftrightarrow \text{ this } = 0 \text{ for every } n\geq 0} x^{n+r}$$

The $n=0$ case is : $0 = r(r-1) a_0 + p_0 a_0 r + q_0 a_0$

i.e. the indicial equation

$$0 = r(r-1) + P(0)\, r + Q(0)$$

must hold: *if* there is going to be a solution of the form (*), then it must start with x^r with r a root of this equation.

To go deeper, let's concentrate on a specific example (since the result in general is much more complicated than the recursive stuff we did previously): namely, the Bessel equation

$$x^2 f'' + x f' + (x^2 - \alpha^2) f = 0$$

i.e. $Lf = 0$ with $L = x^2 D^2 + x D + (x^2 - \alpha^2)$, where $\alpha \in \mathbb{R}_{\geq 0}$. First notice that

$$(xD)^2 f = xD(xDf) = xD(xf') \overset{\text{product rule}}{=} x^2 f'' + x f' = (x^2 D^2 + xD) f$$

so that

$$L = (xD)^2 + (x^2 - \alpha^2).$$

The indicial equation is $0 = r(r-1) + r - \alpha^2 = r^2 - \alpha^2$ so that we should seek solutions of the form

$$u_1(x) = \sum_{n\geq 0} a_n x^{n+\alpha} \quad \text{and} \quad u_2(x) = \sum_{n\geq 0} b_n x^{n-\alpha} \quad (x>0)$$

with $a_0 = 1$. (For $x \neq 0$ the solutions are $|x|^{\pm\alpha} \sum a_n x^n$; though if $\alpha \in \mathbb{Z}$ we can take $a_0 = (-1)^\alpha$ for the $x<0$ half and get $x^\alpha \sum a_n x^n$ for the whole thing.) For u_1, we get

$$0 = Lu_1 = \sum_{m\geq 0} \Big\{ \overbrace{\underbrace{(m+\alpha)^2}_{\text{from } (xD)^2} - \alpha^2}^{n(n+2\alpha)} \Big\} a_m x^{m+\alpha} + \sum_{m\geq 0} a_m x^{m+\alpha+2}$$

reindex: replace m by $m-2$, sum over $m \geq 2$

$$= 0x^\alpha + (1+2\alpha) a_1 x^{1+\alpha} + \sum_{m\geq 2} \left\{ m(m+2\alpha) a_m + a_{m-2} \right\}$$

$$\Rightarrow a_1 = 0, \quad a_m = \frac{-a_{m-2}}{m(m+2\alpha)} \qquad (\text{remember } \alpha \geq 0)$$

$$\Rightarrow a_{2n+1} = 0 \quad \text{and} \quad a_{2n} = \frac{(-1)^n}{2^{2n}\, n!\, (1+\alpha)(2+\alpha)\cdots(n+\alpha)}$$

$$\Rightarrow u_1(x) = x^\alpha \sum_{n\geq 0} \frac{(-x^2/4)^n}{n!\,(1+\alpha)\cdots(n+\alpha)}.$$

The same approach works for $r = -\alpha$ and u_2 <u>provided</u> <u>$\alpha \notin \mathbb{Z}$</u>:

$$u_2(x) = x^{-\alpha} \sum_{n\geq 0} \frac{(-x^2/4)^n}{n!\,(1-\alpha)\cdots(n-\alpha)}.$$

<u>But what if α is an integer?</u> Then there is clearly a problem, because for $n = \alpha, \alpha+1, \alpha+2, \ldots$ the denominator blows up — basically, there is no solution to $Lu_2 = 0$ of the desired form.

So how should we modify our "ansatz" for the second solution? First, a bit on the Gamma function: remember that for $s > 0$ we may define

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\, dt .$$

To define Γ on $\mathbb{R}_{<0} \setminus \mathbb{Z}_{<0}$, we can use the functional equation $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ which comes from

$$s\Gamma(s) = \lim_{a\to 0^+} \lim_{b\to\infty} \int_a^b \underbrace{e^{-t}}_{u} \underbrace{s t^{s-1}\, dt}_{dv} \underset{\substack{\uparrow \\ \text{parts}}}{=} \overbrace{\lim_{a\to 0} \lim_{b\to\infty} e^{-t} t^s}^{0} + \lim_{a\to 0} \lim_{b\to\infty} \int_a^b t^s e^{-t}\, dt$$

$$= \Gamma(s+1).$$

This also implies that $\Gamma(n+1) = n!\, \Gamma(1) = n!$ and

$$\uparrow \int_0^\infty e^{-t}\, dt = 1$$

$\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} = (n+\alpha)\cdots(2+\alpha)(1+\alpha)$ (why?), so that

$u_1(x) = x^\alpha \sum_{n\geq 0} \frac{\Gamma(\alpha+1)\,(-1)^n}{\Gamma(n+1)\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n}$. Dividing this by $2^\alpha \Gamma(\alpha+1)$ gives $J_\alpha(x) := \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}$, the Bessel function of the first kind of order α. (These functions are ubiquitous in physics & engineering; e.g., like the Legendre functions they show up in solutions to the steady-state heat equation, but in cylindrical settings rather than spherical ones.) If $\alpha \notin \mathbb{Z}$, $J_{-\alpha}$ gives the 2^{nd} solution.

Going back to the case where $\alpha \in \mathbb{Z}$, let's now consider $\alpha = 0$: then our first solution reads

$$J_0(x) = \sum_{n\geq 0} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n} = \sum_{n\geq 0} \frac{(-x^2/4)^n}{\Gamma(n+1)^2}.$$

Now we take a "Frobenius deformation", which means to formally modify J_0 by replacing n everywhere by $n+\varepsilon$ ($\varepsilon > 0$ small):

$$J_0^\varepsilon(x) := \sum_{n\geq 0} \frac{\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}(-x^2/4)^{n+\varepsilon}$$

I am also throwing in this harmless constant.

Allowing $L = (xD)^2 + x^2$ to operate on this, we get

$$LJ_0^\varepsilon = \sum_{n\geq 0}(2n+2\varepsilon)^2\frac{\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}\left(-\frac{x^2}{4}\right)^{n+\varepsilon} + \underbrace{\sum_{n\geq 0}\frac{(-4)\,\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}\left(-\frac{x^2}{4}\right)^{n+1+\varepsilon}}$$

reindex: $n \mapsto n-1$, notice that $\dfrac{1}{\Gamma(n+\varepsilon)} = \dfrac{n+\varepsilon}{\Gamma(n+\varepsilon+1)}$

$$= 4\sum_{n\geq 0}(n+\varepsilon)^2\frac{\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}\left(-\frac{x^2}{4}\right)^{n+\varepsilon} - 4\sum_{n\geq 1}(n+\varepsilon)^2\frac{\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}\left(-\frac{x^2}{4}\right)^{n+\varepsilon}$$

$$= 4\varepsilon^2(-x^2/4)^\varepsilon.$$

Now (partial-) differentiate both sides w.r.t. ε:

$$L\left(\frac{\partial}{\partial\varepsilon}J_0^\varepsilon\right) = \frac{\partial}{\partial\varepsilon}LJ_0^\varepsilon = \frac{\partial}{\partial\varepsilon}4\varepsilon^2(-x^2/4)^\varepsilon = 8\varepsilon(-x^2/4)^\varepsilon + O(\varepsilon^2)$$

and let $\varepsilon \to 0$:

$$L\left(\frac{\partial J_0^\varepsilon}{\partial\varepsilon}\bigg|_{\varepsilon=0}\right) = 0.$$ So $\dfrac{\partial J_0^\varepsilon}{\partial\varepsilon}\bigg|_{\varepsilon=0}$ is another solution!

Let's calculate it.

Going back to Gamma functions for a moment:

$$\Gamma(z+1) = z\,\Gamma(z) \xrightarrow{d/dz} \Gamma'(z+1) = z\,\Gamma'(z) + \Gamma(z)$$

$$\xrightarrow[\Gamma(z+1) = z\Gamma(z)]{\text{divide by}} \underbrace{\frac{\Gamma'(z+1)}{\Gamma(z+1)}}_{\psi(z+1)} = \underbrace{\frac{\Gamma'(z)}{\Gamma(z)}}_{\psi(z)} + \frac{1}{z}.$$

"digamma function"

So $\psi(z+1) - \psi(z) = \frac{1}{z} \Longrightarrow \psi(n+\varepsilon+1) - \psi(\varepsilon+1) =$

$$\frac{1}{n+\varepsilon} + \cdots + \frac{1}{2+\varepsilon} + \frac{1}{1+\varepsilon} =: H_n^\varepsilon$$

which when $\varepsilon \to 0$ become the harmonic numbers $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$.

Now $\frac{\partial}{\partial\varepsilon} \log\left(\frac{\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}\left(\frac{-x^2}{4}\right)^{n+\varepsilon}\right) = \frac{\partial}{\partial\varepsilon}\left(2\log\Gamma(\varepsilon+1) - 2\log\Gamma(n+\varepsilon+1) + (n+\varepsilon)\log\left(-x^2/4\right)\right)$

$$= 2\left(\psi(\varepsilon+1) - \psi(n+\varepsilon+1)\right) + \log\left(-x^2/4\right) = -2H_n^\varepsilon + \log\left(-x^2/4\right)$$

$$\Longrightarrow \frac{\partial}{\partial\varepsilon} \frac{\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}\left(\frac{-x^2}{4}\right)^{n+\varepsilon} = \left(-2H_n^\varepsilon + \log\left(-x^2/4\right)\right)\frac{\Gamma(\varepsilon+1)^2}{\Gamma(n+\varepsilon+1)^2}\left(\frac{-x^2}{4}\right)^{n+\varepsilon}$$

$$\Longrightarrow \frac{\partial}{\partial\varepsilon}\left(\quad " \quad\right)\Big|_{\varepsilon=0} = \left(-2H_n + \log\left(\frac{-x^2}{4}\right)\right)\frac{(-x^2/4)^n}{(n!)^2}$$

$$\Longrightarrow \frac{\partial J_0^\varepsilon}{\partial\varepsilon}\Big|_{\varepsilon=0} = \sum_{n\geq 0}\left(-2H_n + \log\left(\frac{-x^2}{4}\right)\right)\frac{(-x^2/4)^n}{(n!)^2},$$ from which

we can subtract off the $\log\left(-\frac{1}{4}\right)\sum \frac{(-x^2/4)^n}{(n!)^2} = \text{Const} \times J_0$. Dividing the result by 2, we get $u_2 =$

$$K_0(x) := \log(x)\,J_0(x) + \sum_{n\geq 0}(-1)^{n+1}\frac{H_n}{(n!)^2}\left(\frac{x}{2}\right)^{2n},$$

the Bessel function of the 2nd kind of order 0. (There is a $K_p(x)$ for every $p \in \mathbb{N}$ as well, see Apostol.)

We conclude by stating Frobenius's theorem on solutions to $x^2 f'' + x P(x) f' + Q(x) f = 0$.

If the roots of the indicial equation are $r_1 \geq r_2$, then we always have a solution (on $(0, \rho)$)

$$u_1(x) = x^{r_1} \sum_{n \geq 0} a_n x^n \quad \text{with } a_0 \neq 0;$$

while there is always an independent solution of the form

$$u_2(x) = \begin{cases} x^{r_2} \sum_{n \geq 0} b_n x^n \quad (\text{with } b_0 \neq 0), & r_1 - r_2 \notin \mathbb{Z} \\ x^{r_2} \sum_{n \geq 0} b_n x^n + C\, u_1(x) \log(x), & r_1 - r_2 \in \mathbb{Z}. \end{cases}$$

Here C may be zero (e.g. for the Bessel functions with $r_1 = -r_2 = \alpha \in \frac{1}{2} + \mathbb{Z}$), or not (e.g. for the Bessel functions with $\alpha \in \mathbb{Z}$, as we just saw for $\alpha = 0$).

But the essence of the Frobenius "method" is the ε-deformation approach described above (but not mentioned in Apostol).