Lecture 21: Systems of Linear Diff. Eqns.

In order to prove the main existence & uniqueness theorem for solutions of a single linear DE of order n, we will need to reduce it to a system of n first-order linear DEs. These are interesting in their own right, and today we'll introduce them in just the constant coefficient case.

The systems just mentioned can be expressed in the form

$$(1) \qquad \vec{x}\,'(t) = A\,\vec{x}(t)$$

with A an $n \times n$ matrix. We look at how eigen-stuff allows us to solve this equation, at least when A is diagonalizable, and some applications.

But first, how might one get from the order-n equation with constant coefficients

$$0 = f^{(n)}(t) + a_1 f^{(n-1)}(t) + \cdots + a_n f(t)$$

to something like (1)? You must set $\vec{x}(t) = \begin{pmatrix} f(t) \\ f'(t) \\ \vdots \\ f^{(n-1)}(t) \end{pmatrix}$

so that

$$\vec{x}'(t) = \begin{pmatrix} f'(t) \\ \vdots \\ f^{(n-1)}(t) \\ f^{(n)}(t) \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}}_{=:\, A} \begin{pmatrix} f(t) \\ f'(t) \\ \vdots \\ f^{(n-1)}(t) \end{pmatrix} = A\vec{x}(t)$$

We'll return to this point in subsequent lectures.

Real-diagonalizable case

Ex 1/ Give the general solution to

$$\frac{dx_1}{dt} = x_1(t) + 2x_2(t)$$

$$\frac{dx_2}{dt} = -x_1(t) + 4x_2(t).$$

The corresponding matrix equation is

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad \left(A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix},\ \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)$$

$$= P_{\mathcal{B}} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} P_{\mathcal{B}}^{-1} \vec{x} \qquad \left(P_{\mathcal{B}} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

If $\vec{x}(t) = c_1(t)\vec{v}_1 + c_2(t)\vec{v}_2 = P_{\mathcal{B}}\vec{c}(t)$, then also

$$\frac{d\vec{x}}{dt} = P_{\mathcal{B}} \frac{d\vec{c}}{dt} \qquad \text{(entries of } P_{\mathcal{B}} \text{ being constant)}$$

and so the system becomes

$$P_{\mathcal{B}} \frac{d\vec{c}}{dt} = P_{\mathcal{B}} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \vec{c}(t)$$

or $$\frac{d\vec{c}}{dt} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \vec{c}(t).$$

This is just the two equations

$$\begin{cases} \dfrac{dc_1}{dt}(t) = 2\, c_1(t) \\[2ex] \dfrac{dc_2}{dt}(t) = 3\, c_2(t) \end{cases}$$

(i.e. we have decoupled the system), which have solutions

$$\begin{cases} c_1(t) = c_1(0)\, e^{2t} \\[2ex] c_2(t) = c_2(0)\, e^{3t} \end{cases}$$

$$\Rightarrow \vec{x}(t) = c_1(0) e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2(0)\, e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad //$$

So here is the general method suggested by Example 1. Suppose that $A = P_B D P_B^{-1}$, with $P_B = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix}$ and $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. Then

$$\vec{x}'(t) = P_B D P_B^{-1} \vec{x}(t) \implies P_B^{-1} \vec{x}'(t) = D P_B^{-1} \vec{x}(t)$$

(and setting $\vec{y}(t) = P_B^{-1} \vec{x}(t)$, $\vec{c} = \vec{y}(0) = P_B^{-1} \vec{x}(0)$)

$$\implies \vec{y}'(t) = D \vec{y}(t)$$

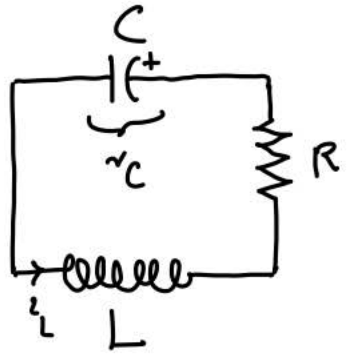
$$\implies y_j'(t) = \lambda_j y_j(t) \quad (\text{for each } j = 1, \ldots, n).$$

$$\implies y_j(t) = c_j e^{\lambda_j t} \quad (\quad \text{"} \quad)$$

This yields the general solution to (1) in the real-diagonalizable case:

(2) $$\vec{x}(t) = P_B \vec{y}(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \vec{v}_j .$$

Ex 2/ Consider the electrical circuit:



i_L = current thru inductor
v_C = charge on capacitor
L = inductance
C = capacitance
R = resistance

This satisfies the equations $\begin{cases} i_L' = -\frac{R}{L} i_L - \frac{1}{L} v_C \\ v_C' = \frac{1}{C} i_L \end{cases}$, which

we can express as $\vec{x}'(t) = A\vec{x}(t)$ with $A = \begin{pmatrix} -R/L & -1/L \\ 1/C & 0 \end{pmatrix}$, $\vec{x}(t) = \begin{pmatrix} i_L \\ v_C \end{pmatrix}$.

Suppose $R = 5/2$, $L = 1 = C$; then $A = \begin{pmatrix} -5/2 & -1 \\ 1 & 0 \end{pmatrix}$, and

$\det(A - \lambda I_2) = \begin{vmatrix} -\frac{5}{2} - \lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + \frac{5}{2}\lambda + 1$ has roots $\lambda_1 = -\frac{1}{2}$, $\lambda_2 = -2$.

For $\vec{v}_1$, $\mathrm{Nul}(A + \frac{1}{2}I) = \mathrm{Nul}\begin{pmatrix} -2 & -1 \\ 1 & 1/2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{\vec{v}_1} \right\}$.

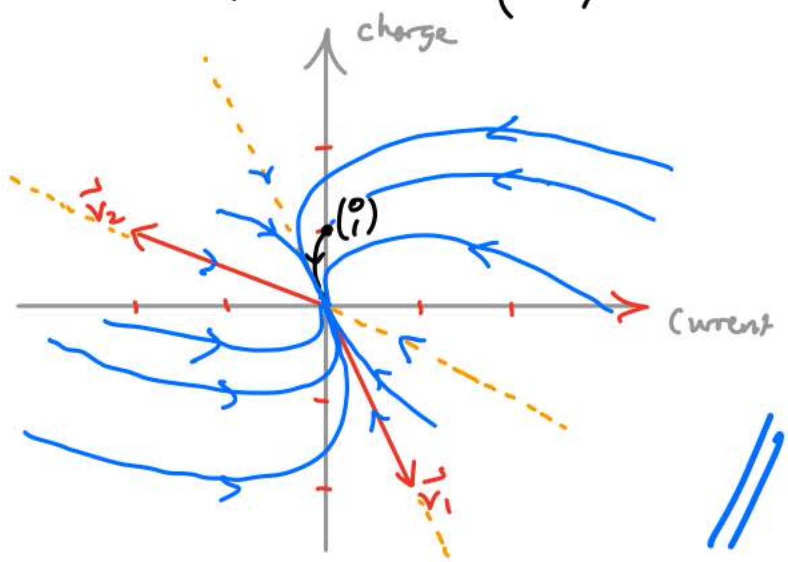
For $\vec{v}_2$, $\mathrm{Nul}(A + 2I) = \mathrm{Nul}\begin{pmatrix} -1/2 & -1 \\ 1 & 2 \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \underbrace{\begin{pmatrix} -2 \\ 1 \end{pmatrix}}_{\vec{v}_2} \right\}$.

If $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (start w/ charge on capacitor but no current),

then $\vec{c} = P_{\mathcal{B}}^{-1}\vec{x}(0) = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1/3 \end{pmatrix}$.

Plugging in to (2) gives $\vec{x}(t) = -\frac{2}{3} e^{-t/2} \begin{pmatrix} 1 \\ -2 \end{pmatrix} - \frac{1}{3} e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

The origin is an attractor, as it should be: the voltage is dissipating through the resistor and inductor.



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Complex-diagonalizable case

We'll work this out for 2×2 matrices only. Suppose A has an eigenvalue $\lambda = a + ib$ with corresponding eigenvector $\vec{v} = \vec{u} + i\vec{w}$ ($A, a, b, \vec{u}, \vec{w}$ all real). Write

$$c_1 \vec{v} + c_2 \overline{\vec{v}} = \vec{x}(0) \underset{\substack{\uparrow \\ \vec{x}(t) \text{ real}}}{=} \overline{\vec{x}(0)} = \overline{c_1}\,\overline{\vec{v}} + \overline{c_2}\,\vec{v}$$

$\Rightarrow c_2 = \overline{c_1}$. Set $c_1 = \frac{\alpha + i\beta}{2}$, $c_2 = \frac{\alpha - i\beta}{2}$ ($\alpha, \beta \in \mathbb{R}$).

We can again use (2) to solve $\vec{x}'(t) = A\vec{x}(t)$:

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\overline{\lambda} t}\overline{\vec{v}}$$

$$= \frac{\alpha + i\beta}{2} e^{(a+ib)t}(\vec{u} + i\vec{w}) + \frac{\alpha - i\beta}{2} e^{(a-ib)t}(\vec{u} - i\vec{w})$$

$$= \frac{e^{at}}{2}\left\{ \underbrace{\left(e^{ibt} + e^{-ibt}\right)}_{2\cos bt}(\alpha\vec{u} - \beta\vec{w}) + \underbrace{i\left(e^{ibt} - e^{-ibt}\right)}_{-2\sin bt}(\beta\vec{u} + \alpha\vec{w}) \right\}$$

$e^{iz} = \cos z + i \sin z \Longrightarrow$ (for $z \in \mathbb{R}$)

$$(3) \qquad = \alpha \cdot e^{at}\left\{(\cos bt)\vec{u} - (\sin bt)\vec{w}\right\} - \beta \cdot e^{at}\left\{(\sin bt)\vec{u} + (\cos bt)\vec{w}\right\}$$

The next example is motivated by the following question: If we change the resistance in Example 2, what happens?

It turns out that the discriminant in the quadratic formula for λ is negative ($\Rightarrow$ complex roots) if $R^2 < 4\frac{L}{C}$, i.e. if the resistance is small. The capacitor will still have to "unload," but since current through the inductor has "inertia", with very little resistance that inertia will cause the current and charge to oscillate.

Ex 3/ Decrease the "R" (in Example 2) to 1, so $A = \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}$.

Then the characteristic polynomial is $\lambda^2 + \lambda + 1$, with roots $\lambda_\pm = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ $\left(\Rightarrow a = -\frac{1}{2},\ b = \frac{\sqrt{3}}{2}\right)$. We have

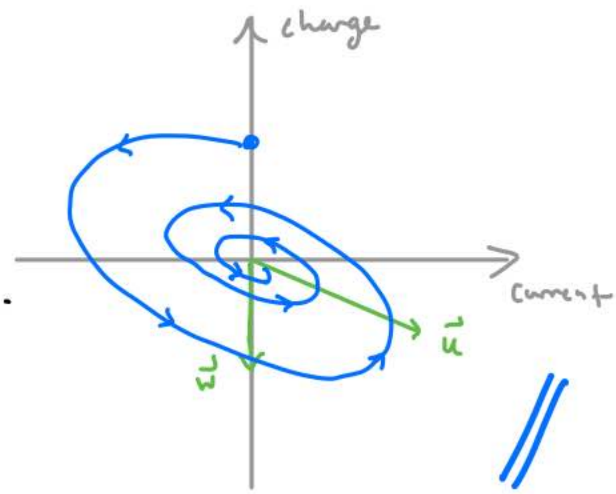
$$\text{Nul}\left(A - (a+ib)\mathbb{I}\right) = \text{Nul}\begin{pmatrix} -\frac{1}{2} - i\frac{\sqrt{3}}{2} & -1 \\ 1 & \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} = \text{span}\left\{\underbrace{\begin{pmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix}}_{\vec{v}}\right\}$$

$\Rightarrow \vec{v} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 0 \\ -\sqrt{3}/2 \end{pmatrix}$. Moreover, $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{3}}\vec{v} - \frac{i}{\sqrt{3}}\bar{\vec{v}}$

$\Rightarrow \alpha = 0,\ \beta = \frac{2}{\sqrt{3}}$. By (3),

$$\vec{x}(t) = \frac{-2}{\sqrt{3}} e^{-t/2}\left\{\sin\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + \cos\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \end{pmatrix}\right\}$$

$$= \frac{2}{\sqrt{3}} e^{-t/2}\left\{\sin\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} -1 \\ \frac{1}{2} \end{pmatrix} + \cos\left(\frac{\sqrt{3}}{2}t\right)\begin{pmatrix} 0 \\ \sqrt{3}/2 \end{pmatrix}\right\}.$$



Non-diagonalizable case

What happens in a case like

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where A is not diagonalizable if $a \neq 0$? (The characteristic polynomial is $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$, but $E_1 = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ is only 1-dimensional.) Well, $e^{1t}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^t\\0\end{pmatrix}$ is a solution just like in the above cases, but how do we get a <u>second</u> solution?

Since this is the end of the lecture, let's just guess:

$$\begin{pmatrix}ate^t\\e^t\end{pmatrix}' = \begin{pmatrix}ate^t + ae^t\\e^t\end{pmatrix} = \begin{pmatrix}1 & a\\0 & 1\end{pmatrix}\begin{pmatrix}ate^t\\e^t\end{pmatrix}$$

works, so that's the second solution.