Lecture 22: Exponentiating Matrices

Recall the basic problem introduced in Lecture 21: solving a system of n first-order DEs

$$\begin{cases} x_1'(t) = a_{11}x_1(t) + \cdots + a_{1n}x_n(t) \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ x_n'(t) = a_{n1}x_1(t) + \cdots + a_{nn}x_n(t) \end{cases} \underset{\text{equivalently}}{\overset{\text{or}}{\iff}} \overset{(*)}{\boxed{\vec{x}'(t) = A\vec{x}(t)}}$$

$$(\text{where } A = [a_{ij}]).$$

We devised a method in the case that $A = P\overset{\text{diag}(\lambda_1,\dots,\lambda_n)}{D}P^{-1}$ (i.e. A is diagonalizable), by substituting $\vec{x}(t) = P\vec{y}(t)$

$$\Rightarrow \quad P\vec{y}'(t) = \vec{x}'(t) = AP\vec{y}(t) = PDP^{-1}P\vec{y}(t) = PD\vec{y}(t)$$

$$\Rightarrow \quad \vec{y}'(t) = D\vec{y}(t) \longrightarrow y_j'(t) = \lambda_j y_j(t) \Rightarrow y_j(t) = c_j e^{\lambda_j t}$$

(where $\vec{c} = P^{-1}\vec{x}(0)$). But what about the "general A" case?

What if we could simply solve $(*)$ by writing

$$\vec{x}(t) := e^{tA}\vec{x}(0) \quad \Rightarrow \quad \vec{x}'(t) = Ae^{tA}\vec{x}(0) = A\vec{x}(t) \; ?$$

To do this, we need to introduce calculus for matrix-valued functions $M = M(t) = [m_{ij}(t)]$ (that is, n^2 functions if M is $n \times n$). We define $M' = M'(t) = [m_{ij}'(t)]$ and notice that $(M+N)' = M' + N'$ and $(MN)' = M'N + MN'$ (why?).

We'll also need series of matrices: given a sequence

$\{C_k\}_{k\geq 0}$ with $C_k = [c_{ij}^{(k)}]$, define $\sum_{k=0}^{\infty} C_k := \left[\sum_k c_{ij}^{(k)}\right]$

(that is, "convergence" means that every matrix entry converges).

Lemma: If $\sum \|C_k\|$ converges (the norm of a matrix $\|A\| := \sum_i \sum_j |a_{ij}|$), then $\sum C_k$ converges.

Proof: $|c_{ij}^{(k)}| \leq \|C_k\| \Rightarrow \sum |c_{ij}^{(k)}| \leq \sum \|C_k\|$. So if $\sum \|C_k\|$ converges, $\sum c_{ij}^{(k)}$ converges absolutely ($\Rightarrow$ converges). $\square$

Ex/ First note that $\|AB\| = \sum_i \sum_k \left|\sum_j a_{ij} b_{jk}\right| \leq \sum_i \sum_j |a_{ij}| \sum_k |b_{jk}|$

$\leq \sum_i \sum_j |a_{ij}| \, \|B\| \leq \|A\| \|B\|$, so $\|A^k\| \leq \|A\|^k$.

Taking $C_k = \frac{1}{k!} A^k$, $\|C_k\| = \frac{1}{k!}\|A^k\| \leq \frac{1}{k!}\|A\|^k \longrightarrow$

$\sum \|C_k\| \leq \sum \frac{\|A\|^k}{k!} = e^{\|A\|} \underset{\text{Lemma}}{\Rightarrow} \sum C_k = \sum \frac{1}{k!} A^k$ converges.

Define $e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$. (The 0th term is by defn. I.)

e.g. $e^0 = I$

$$e^{tA} = \sum_k \frac{(tA)^k}{k!} = \sum_k \frac{t^k}{k!} A^k$$

if $A = D = \text{diag}(\lambda_1, \dots, \lambda_n)$, $e^{tD} = \sum \frac{t^k}{k!} \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}$. //

Theorem 1: The matrix differential equation

$$M'(t) = A\,M(t) \quad \text{with initial cond. } M(0) = B$$

has the unique solution $M(t) = e^{tA} B$.

Note that $A^{k+1} = AA^k = A^kA \Rightarrow Ae^{tA} = e^{tA}A$.

<u>Proof</u>: Write $A^k = [c_{ij}^{(k)}]$. Then

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \left[\sum_{k=0}^{\infty} \frac{t^k}{k!} c_{ij}^{(k)}\right] \quad \text{while}$$

$$(e^{tA})' = \left[\sum_{k=1}^{\infty} \frac{\cancel{k}\, t^{k-1}}{\underset{(k-1)!}{\cancel{k!}}} c_{ij}^{(k)}\right] \underset{\substack{\uparrow \\ \text{replace } k \\ \text{by } k+1}}{=} \left[\sum_{k=0}^{\infty} \frac{t^k}{k!} c_{ij}^{(k+1)}\right] = \sum \frac{t^k}{k!} \underbrace{\left[c_{ij}^{(k+1)}\right]}_{A^{k+1}}$$

$$= \sum \frac{t^k}{k!} AA^k = A \sum \frac{t^k}{k!} A^k = Ae^{tA}.$$

So $(e^{tA}B)' = (e^{tA})'B = A(e^{tA}B) \Rightarrow M(t) = e^{tA}B$ gives a solution.

Next write $F(t) := e^{tA}e^{-tA}$, so that

$$F'(t) = (e^{tA})'e^{-tA} + e^{tA}(e^{-tA})' = Ae^{tA}e^{-tA} + e^{tA}(-Ae^{-tA})$$

$$= Ae^{tA}e^{-tA} - Ae^{tA}e^{-tA} = 0 \Rightarrow F(t) \text{ is constant}$$

$$\Rightarrow F(t) = F(0) = e^0e^0 = I \Rightarrow e^{-tA} = \text{inverse of } e^{tA}.$$

Therefore given <u>any</u> solution $M(t)$, putting

$$G(t) := e^{-tA}M(t) \text{ gives } G'(t) = -Ae^{-tA}M(t) + e^{-tA}\underbrace{M'(t)}_{AM(t)} = 0$$

$$\Rightarrow G(t) \text{ constant} \Rightarrow G(t) = G(0) = M(0) = B$$

$$\Rightarrow B = e^{-tA}M(t) \Rightarrow e^{tA}B = e^{tA}e^{-tA}M(t) = M(t). \quad \square$$

<u>Corollary</u>: <u>If</u> $AB = BA$, then $e^{A+B} = e^Ae^B$. (The statement is false if $AB \neq BA$.)

<u>Proof</u>: By induction on k, $A^kB = BA^k$; hence $e^{tA}B = Be^{tA}$. Both $e^{t(A+B)}$ and $e^{tA}e^{tB}$ therefore solve $M'(t) = (A+B)M(t)$, with initial condition $M(0) = I$. Hence they are equal. $\square$

Theorem 2: Given A $n\times n$ matrix, $\vec{b} \in \mathbb{R}^n$ and $a \in \mathbb{R}$, there exists a unique solution to $\vec{x}'(t) = A\vec{x}(t)$ with $\vec{x}(a) = \vec{b}$.

Proof: Suppose $\vec{x}(t)$ is a solution. Then set $\vec{w}(t) := e^{-tA}\vec{x}(t)$, compute

$$\vec{w}'(t) = -Ae^{-tA}\vec{x}(t) + e^{-tA}\underbrace{\vec{x}'(t)}_{A\vec{x}(t)} = -A\vec{w}(t) + A\vec{w}(t) = \vec{0}$$

$\Rightarrow$ $\vec{w}(t)$ is constant

$\Rightarrow$ $\vec{w}(t) = \vec{w}(a) = e^{-aA}\vec{x}(a) = e^{-aA}\vec{b}$

$\Rightarrow$ $\vec{x}(t) = e^{tA}\vec{w}(t) = e^{(t-a)A}\vec{b}$. Check that this works:

$$\vec{x}'(t) = Ae^{(t-a)A}\vec{b} = A\vec{x}(t).$$

□

Corollary: Given $b_1, \dots, b_n$ and $a, a_1, \dots, a_n \in \mathbb{R}$, then exists a unique solution $f(t)$ to

$$\begin{cases} f^{(n)}(t) + a_1 f^{(n-1)}(t) + \dots + a_n f(t) = 0 \\ f(a) = b_1, \ f'(a) = b_2, \ \dots, \ f^{(n-1)}(a) = b_n. \end{cases}$$

Proof: Let $A = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Then $\begin{cases} \vec{x}'(t) = A\vec{x}(t) \\ + \vec{x}(a) = \vec{b} \end{cases} \Leftrightarrow \begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = x_3(t) \\ \vdots \\ x_{n-1}'(t) = x_n(t) \\ x_n'(t) = -a_1x_n(t) - \dots - a_nx_1(t) \end{cases}$ $+ \ \vec{x}(a) = \vec{b}$

$$\underset{\substack{\uparrow \\ \text{writing} \\ x_1(t)=f(t)}}{\Longleftrightarrow} \begin{cases} x_2(t) = f'(t) \\ x_3(t) = f''(t) \\ \quad \vdots \\ x_n(t) = f^{(n-1)}(t) \\ f^{(n)}(t) = -a_1 f^{(n-1)}(t) - \dots - a_n f(t) \end{cases} + \begin{cases} f(a) = b_1 \\ \quad \vdots \\ f^{(n-1)}(a) = b_n \end{cases}.$$

Since the first system has a unique solution, so does the last. $\square$

Great! So how do we compute this thing e^{tA} that solves everything? In general, that's the topic of next lecture, but here are a couple of examples.

- If $A = PDP^{-1}$ (is diagonalizable), then

$$A^k = PD^kP^{-1} \Longrightarrow e^{tA} = \sum \frac{t^k}{k!} A^k = P \sum \frac{t^k}{k!} P^{-1} = Pe^{tD}P^{-1}$$

where $e^{tD} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}$.

Ex/ $$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \underset{P}{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}} \underset{D}{\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}} \underset{P^{-1}}{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}}$$

$$\Longrightarrow e^{tA} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \left(\begin{array}{c|c} 2e^{2t} - e^{3t} & -2e^{2t} + 2e^{3t} \\ \hline e^{2t} - e^{3t} & -e^{2t} + 2e^{3t} \end{array}\right)$$

$\Longrightarrow$ Sol'n to $\vec{x}'(t) = A\vec{x}(t)$ w/ init. conditions $\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\vec{x}(t) = e^{tA} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2e^{2t} - e^{3t} \\ e^{2t} - e^{3t} \end{pmatrix}. \quad //$$

- What if (say) $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$? By induction on k,

$A^k = \begin{pmatrix} 1 & k\alpha \\ 0 & 1 \end{pmatrix}$. So $\left[\text{using } \sum \frac{t^k}{k!} k\alpha = \alpha \sum \frac{t^k}{(k-1)!} = \alpha \sum \frac{t^{k+1}}{k!} = \alpha t e^t \right]$

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{pmatrix} 1 & k\alpha \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^t & \alpha t e^t \\ 0 & e^t \end{pmatrix}$$

$\Rightarrow$ solution with initial condition $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $e^{tA}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$

" " " " $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is $e^{tA}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha t e^t \\ e^t \end{pmatrix}$.