Lecture 24: The existence & uniqueness theorem

We conclude our segment on differential equations by proving the big theorem we've been assuming all along: that given an equation
$$f^{(n)} + P_1(t) f^{(n-1)} + \dots + P_n(t) f = 0 \qquad (*)$$
with $P_1, \dots, P_n$ continuous on an interval I, there exists a unique $f(t)$ solving it with prescribed "initial values" $f(a) = b_1$, $f'(a) = b_2$, ..., $f^{(n-1)}(a) = b_n$. By recasting $(*)$ as the vector equation $\vec{x}'(t) = A(t)\vec{x}(t)$, with
$$\vec{x}(t) = \begin{pmatrix} f(t) \\ f'(t) \\ \vdots \\ f^{(n-1)}(t) \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & \\ 0 & 0 & 1 & 0 & \cdots \\ & \text{O} & & \ddots & \text{O} \\ & & & & 1 \\ -P_n(t) & \cdots & \cdots & \cdots & -P_1(t) \end{pmatrix},$$
this becomes a special case of the following

<u>Theorem</u>: Let $A(t)$ be a continuous matrix-valued function on I, and let $\vec{b} \in \mathbb{R}^n$, $a \in I$. Then there exists a unique solution $\vec{x}(t)$ to
$$\begin{cases} \vec{x}'(t) = A(t)\vec{x}(t) \\ \vec{x}(a) = \vec{b} \end{cases}.$$

(For simplicity, in the proof, we shall take $a = 0$, and work only with $t \geq 0$; to do $t < 0$ replace $\int_0^t$ by $\int_t^0$.)

PROOF: Define "successive approximations" to the solution by

$\vec{x}_0(t) := \vec{b}$

solve $\vec{x}_1'(t) = A(t)\vec{x}_0(t)$: $\quad \vec{x}_1(t) := \vec{b} + \int_0^t A(u)\,\vec{b}\,du$

solve $\vec{x}_2'(t) = A(t)\,\vec{x}_1(t)$: $\quad \vec{x}_2(t) := \vec{b} + \int_0^t A(u)\,\vec{x}_1(u)\,du$

$\vdots \qquad\qquad\qquad\qquad \vdots$

solve $\vec{x}_{k+1}'(t) = A(t)\,\vec{x}_k(t)$: $\quad \vec{x}_{k+1}(t) := \vec{b} + \int_0^t A(u)\,\vec{x}_k(u)\,du$

$\downarrow$

Converge to something?

Aside: Recall the definition of uniform convergence on $[a,b]$ of a series of functions $\sum f_n$ (Lecture 37 of last term):

given $\epsilon > 0$ $\exists N$ s.t. $\left|\sum_{n\geq N} f_n(x)\right| < \epsilon \quad \forall x \in [a,b]$.

Key result 1: If $|f_n| \leq M_n \in \mathbb{R}$ and $\sum M_n$ converges, then $\sum f_n$ converges uniformly.

Key result 2: If $\sum f_n$ converges uniformly, then

- f_n continuous $(\forall n) \implies \sum f_n$ continuous
- $\sum_n \int_a^b f_n\,dx = \int_a^b \left(\sum f_n\right) dx$.

Restrict to any closed subinterval $J \subset I$ containing 0, and write $M := \max_{t\in J} \|A(t)\|$ where we recall that $\|\cdot\|$ means to sum the absolute values of matrix entries. (We shall also apply this "norm" to vectors: in this proof $\left\|\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\right\| = |a_1| + \cdots + |a_n|$.)

Write L for the length of J.

From the formulas above,

$$\vec{x}_1(t) - \vec{x}_0(t) = \int_0^t A(u)\vec{b}\,du \Rightarrow \|x_1 - x_0\| = \left\|\int_0^t A(u)\vec{b}\,du\right\|$$
$$\leq \int_0^t \|A(u)\vec{b}\|\,du \leq \int_0^t \|A(u)\|\,\|\vec{b}\|\,du$$
$$\leq t M \|\vec{b}\|$$

$$\vec{x}_2(t) - \vec{x}_1(t) = \int_0^t A(u)\left(\vec{x}_1(u) - \vec{x}_0(u)\right)du \Rightarrow \|x_2 - x_1\| \leq \int_0^t \|A(u)\|\,\|x_1 - x_0\|\,du$$
$$\leq \int_0^t M \cdot u M \|\vec{b}\|\,du$$
$$= \frac{t^2 M^2}{2}\|\vec{b}\|$$

$\vdots$

$$\vec{x}_{m+1}(t) - \vec{x}_m(t) = \int_0^t A(u)\left(\vec{x}_m(u) - \vec{x}_{m-1}(u)\right)du \Rightarrow \|x_{m+1} - x_m\| \leq \int_0^t M \underbrace{\|x_m - x_{m-1}\|}_{\leq \frac{u^m M^m}{m!}\|\vec{b}\|}\,du$$
$$\leq \frac{M^{m+1}\|\vec{b}\|}{m!}\int_0^t u^m\,du$$

(by induction)

$$= \frac{t^{m+1}M^{m+1}}{(m+1)!}\|\vec{b}\|.$$

$$\Rightarrow \sum_{m=0}^{\infty} \|\vec{x}_{m+1}(t) - \vec{x}_m(t)\| \leq \sum_{m=0}^{\infty} \underbrace{\frac{L^{m+1}M^{m+1}}{(m+1)!}\|\vec{b}\|}_{\text{these are your } \{M_m\}} = \left(e^{LM} - 1\right)\|\vec{b}\| < \infty$$

$$\text{VI}$$

$$\sum_{m=0}^{\infty}\underbrace{\left(x_{m+1,\ell}(t) - x_{m,\ell}(t)\right)}_{\ell^{\text{th}} \text{ entry of each vector}}$$

$$\Rightarrow \sum_{m=0}^{\infty}\underbrace{\left(\vec{x}_{m+1}(t) - \vec{x}_m(t)\right)}_{\text{each is continuous by construction}} \text{ converges uniformly on } J$$

$$\Rightarrow \vec{x}(t) := \vec{b} + \sum_{m=0}^{\infty}\left(\vec{x}_{m+1}(t) - \vec{x}_m(t)\right) = \lim_{k\to\infty}\Big(\underbrace{\vec{x}_0(t) + \sum_{m=0}^{k}\left(\vec{x}_{m+1}(t) - \vec{x}_m(t)\right)}_{\text{telescoping sum}}\Big)$$

$$= \lim_{k\to\infty} \vec{x}_{k+1}(t)$$ exists and is continuous on J (hence on all of I since J was arbitrary).

Moreover, $\vec{x}(t) = \lim_{k\to\infty} \vec{x}_{k+1}(t) = \lim_{k\to\infty}\left(\vec{b} + \int_0^t A(u)\vec{x}_k(u)\,du\right)$

$$= \vec{b} + \lim_{k\to\infty}\int_0^t A(u)\vec{x}_k(u)\,du$$

$$= \vec{b} + \int_0^t A(u)\lim_{k\to\infty}\vec{x}_k(u)\,du$$

$$= \vec{b} + \int_0^t A(u)\vec{x}(u)\,du$$

$\Longrightarrow$ $\vec{x}(t)$ is differentiable, with
FTC

$$\vec{x}\,'(t) = A(t)\vec{x}(t), \quad \text{and} \quad \vec{x}(0) = \vec{b}.$$

So existence is proved! That was the hard part.

Now suppose $\vec{x}(t)$, $\vec{\xi}(t)$ are two solutions. Let N be an upper bound on $\|\vec{x}-\vec{\xi}\|$ for $t \in J$. Then

$$\vec{x}\,' - \vec{\xi}\,' = A(t)(\vec{x}-\vec{\xi}) \Longrightarrow \vec{x}-\vec{\xi} = \int_0^t A(u)\left(\vec{x}(u)-\vec{\xi}(u)\right)du$$

$$\Longrightarrow \|\vec{x}(t)-\vec{\xi}(t)\| \le \int_0^t \|A(u)\|\,\|\vec{x}(u)-\vec{\xi}(u)\|\,du \le \int_0^t M\cdot N\,du = MNt$$

$$\Longrightarrow \|\vec{x}(t)-\vec{\xi}(t)\| \le \int_0^t M\cdot\|\vec{x}(u)-\vec{\xi}(u)\|\,du \le \int_0^t M\cdot MNu\,du = \frac{M^2t^2}{2}N$$

$$\Longrightarrow \underset{\text{(inductively)}}{\cdots} \Longrightarrow \|\vec{x}(t)-\vec{\xi}(t)\| \le \int_0^t M\cdot\frac{M^{m-1}u^{m-1}}{(m-1)!}\,du = \frac{M^m t^m}{m!}N$$

but then $\|\vec{x}(t)-\vec{\xi}(t)\| \le N\dfrac{(ML)^m}{m!} \xrightarrow{m\to\infty} 0 \Longrightarrow \vec{x}(t) = \vec{\xi}(t)$ on J (hence I).

(Recalling that "factorials dominate exponentials") $\square$

Here is a simple example to illustrate the method embedded in the proof: Say we want to solve

$\vec{x}\,'(t) = A\vec{x}(t)$, with A now <u>constant</u>. Then

$$\vec{x}_1 = \vec{b} + \int_0^t A\vec{b}\,du = \vec{b} + tA\vec{b}$$

$$\vec{x}_2 = \vec{b} + \int_0^t A\vec{x}_1(u)\,du = \vec{b} + \int_0^t A(\vec{b} + uA\vec{b})\,du = \vec{b} + tA\vec{b} + \frac{t^2}{2}A^2\vec{b}$$

$$\vdots$$

$$\vec{x}_k = \vec{b} + tA\vec{b} + \frac{t^2}{2}A^2\vec{b} + \cdots + \frac{t^k}{k!}A^k\vec{b} = \sum_{j=0}^{k}\frac{t^j}{j!}A^j\vec{b}$$

$$\Big\downarrow k \to \infty$$

$$\vec{x}(t) = \sum_{j=0}^{\infty}\frac{t^j}{j!}A^j\vec{b} = e^{tA}\vec{b},$$ precisely the solution found before.