Lecture 24: The existence & uniqueness theorem

We conclude our segment on differential equations by proving the big theorem we've been assuming all along: that given an equation

$$f^{(n)} + P_1(t)f^{(n-1)} + \ldots + P_n(t)f = 0 \quad (*)$$

with $P_1, \ldots, P_n$ continuous on an interval $I$, then exists a unique $f(t)$ solving it with prescribed "initial values"

$$f(a) = b_1, \quad f'(a) = b_2, \quad \ldots, \quad f^{(n-1)}(a) = b_{n-1}.$$ 

By rewriting $(*)$ as the vector equation

$$\dot{x}(t) = A(t)x(t),$$

with

$$x(t) = \begin{pmatrix} f(t) \\ f'(t) \\ \vdots \\ f^{(n-1)}(t) \end{pmatrix} \quad \text{and} \quad A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ -P_n(t) & \cdots & \cdots & \cdots & -P_1(t) \end{pmatrix},$$

this becomes a special case of the following

**Theorem:** Let $A(t)$ be a continuous matrix-valued function on $I$, and let $b \in \mathbb{R}^n$, $a \in I$. Then there exists a unique solution $x(t)$ to

$$\begin{cases} \dot{x}(t) = A(t)x(t) \\ x(a) = b \end{cases}$$

(For simplicity, in the proof, we shall take $a=0$, and work only with $t \geq 0$; to do $t < 0$ replace $\int_0^t$ by $\int_t^0$.)
**Proof:** Define "successive approximations" to the solution by

\[
\begin{align*}
    x_{0}(t) & := b \\
    \text{solve } x'_{1}(t) &= A(t) x_{0}(t) : & x_{1}(t) & := b + \int_{0}^{t} A(u) b \, du \\
    \text{solve } x'_{2}(t) &= A(t) x_{1}(t) : & x_{2}(t) & := b + \int_{0}^{t} A(u) x_{1}(u) \, du \\
    \vdots \quad \text{solve } x'_{k}(t) &= A(t) x_{k-1}(t) : & x_{k}(t) & := b + \int_{0}^{t} A(u) x_{k-1}(u) \, du \\
    \downarrow \quad \text{Converge to something?}
\end{align*}
\]

Aside: Recall the definition of uniform convergence on \([a,b]\) of a series of functions \(\sum f_{n}\) (Lecture 37 of last term):

Given \(\epsilon > 0\) \(\exists N\) s.t. \(\left| \sum_{n=1}^{N} f_{n}(x) \right| < \epsilon\) \(\forall x \in [a,b]\).

Key result 1: If \(|f_{n}| \leq M_{n} \in \mathbb{R}\) and \(\sum M_{n}\) converges, then \(\sum f_{n}\) converges uniformly.

Key result 2: If \(\sum f_{n}\) converges uniformly, then

- \(f_{n}\) continuous \((\forall n) \rightarrow \sum f_{n}\) continuous
- \(\sum_{n} \int_{a}^{b} f_{n} \, dx = \int_{a}^{b} (\sum f_{n}) \, dx\).

Restrict to any closed subinterval \(J \subset I\) containing \(0\), and write \(M := \max_{t \in J} \|A(t)\|\) where we recall that \(\| \cdot \|\) means to run the absolute values of matrix entries. (We shall also apply this "norm" to vectors: in this case \(\| (a_{i}) \| = |a_{1}| + \cdots + |a_{n}|\).

Write \(L\) for the length of \(J\).
From the formulas above,

\[ \tilde{x}_1(t) - \tilde{x}_0(t) = \int_0^t A(u) \tilde{e}^u \, du \Rightarrow \| x_1 - x_0 \| = \left\| \int_0^t A(u) \tilde{e}^u \, du \right\| \leq \int_0^t \| A(u) \| \| \tilde{e}^u \| \, du \leq t \| M \| \| \tilde{e}^u \| \]

\[ \tilde{x}_2(t) - \tilde{x}_1(t) = \int_0^t A(u) (\tilde{x}_1(u) - \tilde{x}_0(u)) \, du \Rightarrow \| x_2 - x_0 \| \leq \int_0^t \| A(u) \| \| x_1(u) - x_0(u) \| \, du \leq \int_0^t \| M \cdot u \| \| \tilde{e}^u \| \, du \leq \frac{t^2 M^2}{2} \| \tilde{e}^u \| \]

\[ \vdots \]

\[ \tilde{x}_{m+1}(t) - \tilde{x}_m(t) = \int_0^t A(u) (\tilde{x}_m(u) - \tilde{x}_{m-1}(u)) \, du \Rightarrow \| x_{m+1} - x_m \| \leq \int_0^t \| M \| \| x_m(u) - x_{m-1}(u) \| \, du \]

(by induction)

\[ \sum_{m=0}^{\infty} \| x_{m+1}(t) - x_m(t) \| \leq \sum_{m=0}^{\infty} \left( \frac{M^{m+1}}{(m+1)!} \right) \| \tilde{e}^u \| = (e^{M-1}) \| \tilde{e}^u \| < \infty \]

\[ \sum_{m=0}^{\infty} \frac{\tilde{x}_{m+1}(t) - \tilde{x}_m(t)}{m+1} \text{ entry of each vector} \]

\[ \sum_{m=0}^{\infty} (\tilde{x}_{m+1}(t) - \tilde{x}_m(t)) \text{ converges uniformly on } J \]

\[ \text{each is continuous by construction} \]

\[ \tilde{x}(t) := \tilde{b} + \sum_{m=0}^{L} (\tilde{x}_{m+1}(t) - \tilde{x}_m(t)) = \lim_{k \to \infty} \left( \tilde{x}(t) + \sum_{m=0}^{k} (\tilde{x}_{m+1}(t) - \tilde{x}_m(t)) \right) \]

\[ \text{telescoping sum} \]

\[ \lim_{k \to \infty} \tilde{x}_{k+1}(t) \text{ exists and is continuous on } J \]

(hence on all of } J \text{ since } J \text{ was arbitrary).
Moreover, \( \dot{x}(t) = \lim_{k \to \infty} x_{k,t}(t) = \lim_{k \to \infty} \left( \delta + \int_0^t A(u) \dot{x}_k(u) \, du \right) \)
\[ = \delta + \lim_{k \to \infty} \int_0^t A(u) \dot{x}_k(u) \, du \]
\[ = \delta + \int_0^t A(u) \dot{x}(u) \, du \]

\( \Rightarrow \dot{x}(t) \) is differentiable, with

\[ \dot{x}(t) = A(t) \dot{x}(t), \quad \text{and} \quad \dot{x}(0) = \delta. \]

So existence is proved! That was the hard part.

Now suppose \( \dot{x}(t), \ddot{x}(t) \) are two solutions. Let \( N \) be an upper bound on \( \|x - \ddot{x}\| \) for \( t \in \mathcal{T} \). Then

\( \dot{x}' - \ddot{x}' = A(t) (x - \ddot{x}) \Rightarrow \dot{x} - \ddot{x} = \int_0^t A(u) (\dot{x}(u) - \ddot{x}(u)) \, du \)
\[ \Rightarrow \|\dot{x}(t) - \ddot{x}(t)\| \leq \int_0^t \|A(u)\| \|\dot{x}(u) - \ddot{x}(u)\| \, du \leq \int_0^t MN \, du = MNt \]
\[ \Rightarrow \|\dot{x}(t) - \ddot{x}(t)\| \leq \int_0^t M \|x(u) - \ddot{x}(u)\| \, du \leq \int_0^t MNu \, du = \frac{MNt}{2} \]
\[ \Rightarrow \ldots \Rightarrow \|\dot{x}(t) - \ddot{x}(t)\| \leq \int_0^t M - \frac{M^{m-1}u^{m-1}}{m!} \, du = \frac{M^{m-1}u^m}{m!} \]

(Inductively)

but then \( \|\dot{x}(t) - \ddot{x}(t)\| \leq N \left( \frac{ML}{m!} \right)^m \to 0 \Rightarrow \dot{x}(t) = \ddot{x}(t) \)

on \( \mathcal{T} \) (prove \( I \)).

\( \square \)

Here is a simple example to illustrate the method embedded in the proof: Say we want to solve
\[ \dot{x}'(t) = A \dot{x}(t), \text{ with } A \text{ a new constant.} \]

Then
\[
\begin{align*}
\dot{x}_1 &= b + \int_0^t A \dot{x}_2 \, dt = b + tA \dot{x}_2 \\
\dot{x}_2 &= b + \int_0^t A \dot{x}_1 \, dt = b + \int_0^t A(b + uA \dot{x}_2) \, dt = b + tAb + \frac{t^2}{2!} A^2 \dot{x}_2 \\
& \vdots \\
\dot{x}_n &= b + tAb + \frac{t^2}{2} A^2 b + \cdots + \frac{t^n}{n!} A^n \dot{x}_1 = \sum_{j=0}^{n} \frac{t^j}{j!} A^j b \\
\end{align*}
\]

\[ \lim_{k \to \infty} \]
\[ \dot{x}(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j b = e^{tA} b, \] precisely the solution found before.