Lecture 25: Multivariable functions

Subsets of $\mathbb{R}^n$

We will be interested in functions defined on all of $\mathbb{R}^n$, but also in functions whose domain is a subset of $\mathbb{R}^n$. So here are some basic ideas about such sets.

Given $\vec{a} \in \mathbb{R}^n$, define the open ball about $\vec{a}$ of radius r

$$B(\vec{a};r) := \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}-\vec{a}\| < r\}.$$

(What does this look like for $n=1,2,3$?)

Now if $S \subseteq \mathbb{R}^n$ is any subset, define the interior by

$$\operatorname{int}(S) := \{\vec{a} \in S \mid B(\vec{a};r) \subset S \text{ for some } r>0\}.$$

If $S^c := \mathbb{R}^n \setminus S$ denotes the complement, define the boundary of S by

$$\begin{aligned}\partial S &:= \mathbb{R}^n \setminus (\operatorname{int}(S) \cup \operatorname{int}(S^c)) \\ &= \{\vec{a} \in \mathbb{R}^n \mid B(\vec{a};r)\cap S \text{ and } B(\vec{a};r)\cap S^c \text{ are nonempty } \forall r>0\}.\end{aligned}$$

We say that S is open (in $\mathbb{R}^n$)* if $S = \operatorname{int}(S)$, and closed if S^c is open. Alternatively, S is closed if it contains its boundary ∂S (why?). For an arbitrary subset S, the closure $\overline{S} := S \cup \partial S$.

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* Openness is relative: the interval (a,b) is open in $\mathbb{R}$, but if you include it into $\mathbb{R}^2$ as $(a,b)\times\{0\}$, it is not open in $\mathbb{R}^2$.

Examples of open sets include the open balls above, as well as unions and Cartesian products of open sets like $(a,b) \times (c,d) \subset \mathbb{R}^2$. Also $\mathbb{R}^n$ (as a subset of itself) and $\emptyset$ (the empty set) are open [and closed]. A single point is closed; its complement is open.

Ex/ What sort of set is the domain $\mathrm{Dom}(f)$ for each of $f = x\log(y^2-x)$, $\sqrt{1-x^2-y^2}$, $\dfrac{\sqrt{y-x^2}}{x^2+(y-1)^2}$? //

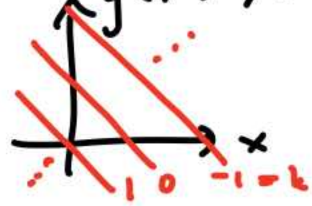
Graphing functions

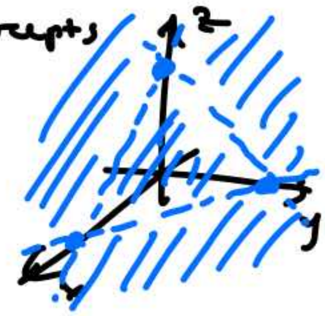
For a function f from a subset of $\mathbb{R}^2$ to $\mathbb{R}$, the graph is the set of solutions to $f(x,y)=z$ in $\mathbb{R}^3$. The level curves $f(x,y)=k$ in $\mathbb{R}^2$ can help to visualize the graph.

k ← fixed real #, e.g. can use integers

- linear functions: $f(x,y) = ax+by+c$ — use the x,y,z intercepts (where it meets the 3 axes)

  e.g. $f(x,y) = 1-x-y$ ⤳ graph $x+y+z=1$ has intercepts $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

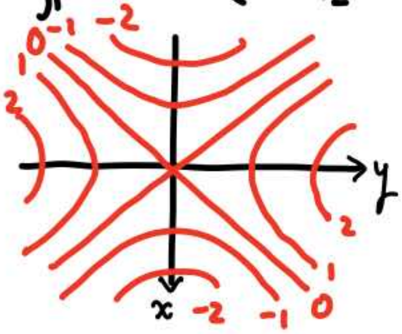
  level curves are 

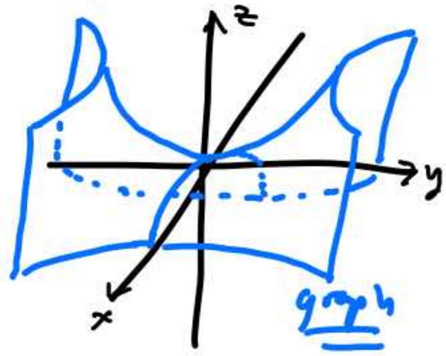


- quadratic functions: $f(x,y) = \underbrace{ax^2+by^2+cxy}+\underbrace{dx+ey+f}$ — can think of as quadratic form + linear part; a linear change of variables turns the first part into $\lambda_1\tilde{x}^2+\lambda_2\tilde{y}^2$, and then completing the square gets rid of the linear terms to give you something of the form $\lambda_1(\tilde{x}-A)^2+\lambda_2(\tilde{y}-B)^2+C=z$ for the graph. This is an elliptic ($\lambda_1/\lambda_2>0$) or hyperbolic ($\lambda_1/\lambda_2<0$) paraboloid, assuming $\lambda_1,\lambda_2\neq 0$.

e.g. $f(x,y)=y^2-x^2$ has

level curves





— you could also try $\frac{1}{4}x^2+y^2-2y$ or xy.

- Other functions related to conic sections: try
$f(x,y)=\sqrt{(x-a)^2+(y-b)^2}$, $\frac{1}{3}\sqrt{36-9x^2-4y^2}$ (what is the domain?), $\dfrac{\sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}$,

Limits

Now consider a set $\mathcal{S}\subseteq\mathbb{R}^n$ and a function

$$\vec{F}:\mathcal{S}\longrightarrow\mathbb{R}^m.$$

Let $\vec{a}\in\overline{\mathcal{S}}$.

<u>Definition 1</u>: $\lim\limits_{\vec{x}\to\vec{a}}\vec{F}(\vec{x})=\vec{L} \iff \lim\limits_{\|\vec{x}-\vec{a}\|\to 0}\|\vec{F}(\vec{x})-\vec{L}\|=0$

$$\iff \forall\epsilon>0\ \exists\delta>0 \text{ s.t. } \underbrace{\vec{x}\in B(\vec{a};\delta)\Rightarrow\vec{F}(\vec{x})\in B(\vec{L};\epsilon)}.$$

e.g. for $n=2$, $m=1$ $(f:(\mathcal{S}\subset\mathbb{R}^2)\to\mathbb{R})$ then this reads

$$\lim_{(x,y)\to(a,b)}f(x,y)=L \iff \forall\epsilon>0\ \exists\delta>0 \text{ s.t.}$$
$$\sqrt{(x-a)^2+(y-b)^2}<\delta \Rightarrow |f(x,y)-L|<\epsilon.$$

<u>Definition 2</u>: $\vec{F}$ is continuous at $\vec{a}$ if

(i) $\vec{F}(\vec{a})$ exists (i.e. $\vec{a} \in S$, not just $\bar{S}$)

(ii) $\lim_{\vec{x}\to\vec{a}} \vec{F}(\vec{x})$ exists

(iii) they are equal. (All 3 must hold!)

The standard limit laws hold: writing "lim" for $\lim_{\vec{x}\to\vec{a}}$,

(a) $\lim(\alpha\vec{F}+\beta\vec{G}) = \alpha\lim\vec{F}+\beta\lim\vec{G}$

(b) $\lim \vec{F}\cdot\vec{G} = \lim\vec{F}\cdot\lim\vec{G}$

(c) $\lim\|\vec{F}\| = \|\lim\vec{F}\|$

(d) $\lim \vec{F}/g = \dfrac{\lim\vec{F}}{\lim g}$ provided $\lim g \neq 0$

(e) for compositions, $(\vec{F}\circ\vec{G})(\vec{x}) := \vec{F}(\vec{G}(\vec{x}))$ has limit $\vec{F}(\lim G(\vec{x}))$ if $\vec{F}$ is continuous at $\lim G(\vec{x})$.

Just as a reminder of how these things work, (b) is proved by writing (if $\lim F = L$, $\lim G = K$)

$$\vec{F}\cdot\vec{G} - \vec{L}\cdot\vec{K} = (\vec{F}-\vec{L})\cdot(\vec{G}-\vec{K}) + \vec{L}\cdot(\vec{G}-\vec{K}) + \vec{K}\cdot(\vec{F}-\vec{L}),$$

taking $\|\cdot\|$ on both sides and applying the triangle inequality.

The main point is that this gives us lots of continuous functions:

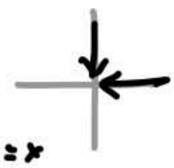
- the identity function $\vec{I}(\vec{x}) := \vec{x}$ is continuous; $\therefore$ so are its components $\vec{e}_k\cdot\vec{I}(\vec{x}) = x_k$ (by (b))
- hence so are polynomials $P(x_1,\dots,x_n) : \mathbb{R}^n \to \mathbb{R}$, and so are rational functions (quotients of polys.) where defined (denominator $\neq 0$), by (a), (b), + (d)
- and so, finally, are functions like $\sin(x^2y)$ or $\log(x^2+y^2)$ or $|x-y|e^{x+y}$ (where defined), by (e)

I'll concentrate on scalar-valued functions from here on out.

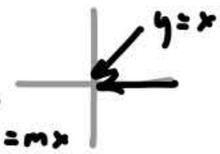
The definition of $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ implies that if you restrict the function to any <u>path</u> approaching $\vec{a}$, you approach the same L (independent of path). For functions of 1 variable, this is just the statement that $x\to a^-$ and $x\to a^+$ need to yield the same limit. But in (say) 2 variables paths could be line segments, parabolic segments, spirals, etc. Since you can't "check all paths," this is mainly useful when 2 paths give <u>different</u> limits, allowing you to conclude that the limit DNE.

Ex/ Show $\lim_{(x,y)\to(0,0)}$ doesn't exist for

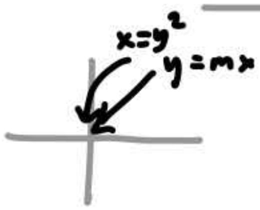
- $\dfrac{x^2-y^2}{x^2+y^2}$ try the paths



- $\dfrac{xy}{x^2+y^2}$



- $\dfrac{xy^2}{x^2+y^4}$



Actually it's interesting to write the first one in polar coordinates: $\dfrac{r^2(\cos^2\theta-\sin^2\theta)}{r^2} = \cos^2\theta-\sin^2\theta$ ~~> see that the limit on a straight path into $(0,0)$ is determined by the angle of approach and can be any number between -1 & 1. //

Since $r\to 0^+$ is <u>equivalent</u> to $(x,y)\to(0,0)$, this suggests a way to prove some limits <u>do</u> exist:

Ex / a $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{r\to0^+} \frac{3(r^2\cos^2\theta)(r\sin\theta)}{r^2}$

$$= \lim_{r\to0^+} 3r\cos^2\theta\sin\theta$$

$$= 0$$ " is a bit heuristic,

but essentially correct. The completely correct way to do this is to say that on $B(\vec{0};\delta)$ (i.e. $r<\delta$)

$$\left|\frac{3x^2y}{x^2+y^2}\right| < 3\delta\,|\cos^2\theta\sin\theta| \le 3\delta.$$

(So taking $\delta = \epsilon/3$ we're done.) Alternatively, you could use the Squeeze theorem and write

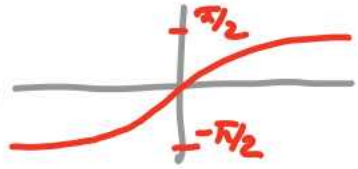
$$0 \le \frac{3x^2|y|}{x^2+y^2} \le 3|y| \;;$$

since $(x,y)\to0$ forces $|y|\to0$, done.

Note that as a result,

$$f(x,y) := \begin{cases} \dfrac{3x^2y}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

is a continuous on $\mathbb{R}^2$. //



Ex / arctan is a continuous function on $\mathbb{R}$

What about $f(x,y) = \arctan(y/x)$ — can we extend this to all of $\mathbb{R}^2$ as a continuous function by "filling in" values along $x=0$?

Fix $y\neq0$ and look at $x\to0^+$ vs. $x\to0^-$.

For $(0,0)$, use $\begin{cases} x=t \\ y=t^2 \end{cases}$ and $\begin{cases} x=t^2 \\ y=t \end{cases}$ $(t\to0^+)$. Answer is no. //