Lecture 26: Partial Derivatives

Given a function \( f : \mathbb{R} \to \mathbb{R} \) defined on a set \( \mathbb{R} \subseteq \mathbb{R} \), the idea of partial derivatives is to hold all variables but one — say, \( x_k \) — constant, and differentiate with respect to \( x_k \) to get \( \frac{\partial f}{\partial x_k} \) or \( D_k f \). If \( n = 2 \), we often write \( (x, y) \) instead of \( x = (x_1, x_2) \) and the geometric idea is to slice the graph \( z = f(x, y) \) by the plane \( y = b \), then compute the slope of \( z = f(x, b) \) at \( x = a \):

\[
\frac{df}{dx}(a, b) := \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}.
\]

More generally,

\[
\frac{df}{dx} := \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \quad \text{and} \quad \frac{df}{dy} := \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}.
\]

Ex 1/ \( f(x, y) = \sin \left( \frac{x}{1+y} \right) \) \( \Rightarrow \frac{df}{dx} = \frac{1}{1+y} \cos \left( \frac{x}{1+y} \right) \) and

\[
\frac{df}{dy} = \left( \frac{1}{1+y} \right) \cos \left( \frac{x}{1+y} \right) = \frac{-x}{(1+y)^2} \cos \left( \frac{x}{1+y} \right).
\]

Ex 2/ \( f(x, y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2} & , \quad (x, y) \neq (0, 0) \\ 0 & , \quad (x, y) = (0, 0) \end{cases} \). We showed last time that this was not continuous at \( (0,0) \) (because limits along paths \( x = y^2 \) and \( y = x^2 \) disagree). But the
partial dx exist: restricting to \( y = 0 \) and differentiating w.r.t. \( x \) gives \( \frac{df}{dx} \bigg|_{y=0} = \frac{d}{dx} f(x,0) \bigg|_{y=0} = \frac{d}{dx} 0 \bigg|_{y=0} = 0 \); and restricting to \( x = 0 \) gives \( \frac{df}{dy} \bigg|_{x=0} = \frac{d}{dy} f(0,y) \bigg|_{x=0} = \frac{d}{dy} 0 = 0 \).

I have drawn a few of the linear paths into \((0,0)\) that gave limits = 0.

There are essentially 3 cases to consider:

- Partially exist and are continuous at \((a,b) \Rightarrow t = f(x,y)\) is well-approximated by a "tangent plane" at \((a,b, f(a,b))\)
- Partially don't exist at \((a,b) \Rightarrow something\)
- Partially exist but are not continuous at \((a,b)\)

- The weird in-between case we find ourselves in above (Ex.2)

We don't consider \( f(x,y) \) "differentiable" at \((a,b)\) in the 2nd or 3rd cases.

Having done the overview, let's turn to definitions, again in the general case of \( \mathbb{R}^n \ni \hat{x} \to f(\hat{x}) \to \mathbb{R} \), with \( \hat{a} \in \text{int}(\mathcal{B}) \).
Derivative with respect to a vector \( \mathbf{y} \in \mathbb{R}^n \):

\[
f'(\mathbf{a}; \mathbf{y}) = \lim_{{h \to 0}} \frac{f(\mathbf{a} + h\mathbf{y}) - f(\mathbf{a})}{h}
\]

- \( f'(\mathbf{a}; \mathbf{0}) = 0 \)
- \( f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{x} \) linear \( \Rightarrow f'(\mathbf{a}; \mathbf{y}) = \lim_{{h \to 0}} \frac{f(\mathbf{a} + h\mathbf{y}) - f(\mathbf{a})}{h} = f(\mathbf{a}) = f(\mathbf{y}) \)
- \( g(t) = f(\mathbf{a} + t\mathbf{y}) \Rightarrow g'(t) = f'(\mathbf{a} + t\mathbf{y}; \mathbf{y}) \)
  
  (e.g. if \( f(\mathbf{x}) = \|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x} \), \( f'(\mathbf{a}; \mathbf{y}) = g'(0) = \frac{d}{dt} (\mathbf{a} + t\mathbf{y}) \cdot (\mathbf{a} + t\mathbf{y}) \bigg|_{{t=0}} = 2 \mathbf{a} \cdot \mathbf{y} \))
- Mean value theorem for \( g(t) \) on \([0, 1]\) \( \Rightarrow f(\mathbf{a} + t\mathbf{y}) - f(\mathbf{a}) = f'(\mathbf{a} + t_0\mathbf{y}; \mathbf{y}) \) for some \( t_0 \in (0, 1) \).
- This is called the directional derivative \( D_{\mathbf{y}} f \) if \( \mathbf{y} = \mathbf{a} \) is a unit vector
- Partial derivatives \( \frac{\partial f}{\partial \mathbf{u}} = D_{\mathbf{u}} f(\mathbf{x}) := D_{\mathbf{u}} f(\mathbf{a}) = \lim_{{h \to 0}} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h} \)
  
  are a special case of directional derivatives.

**Ex 3**

If \( f \) as in Ex. 2, \( \mathbf{y} = (\alpha, \beta) \), \( \mathbf{a} = (0, 0) \)

\[ g(t) = f(\alpha t, \beta t) = \frac{\alpha \beta^2 t}{\alpha^2 + \beta^4 t^2} \]

\[ f'(\mathbf{0}; \mathbf{y}) = g'(0) = \frac{\alpha \beta^2 (2\beta^2 - 2\beta^4 t^2)}{(\alpha^2 + \beta^4 t^2)^2} \bigg|_{{t=0}} = \frac{\beta^2}{\alpha} \quad \text{if} \quad \alpha \neq 0 \]

(or 0 if \( \alpha = 0 \), since then \( g(t) \equiv 0 \))

\[ \Rightarrow \text{all the directional derivatives exist at } (0, 0) \]

\[ \|\mathbf{y}\| = 1 \Rightarrow \alpha^2 + \beta^2 = 1 \], or rather \( \alpha = \cos \theta, \beta = \sin \theta \)

with value \( \frac{\sin^2 \theta}{\cos \theta} \) if \( \cos \theta \neq 0 \), and 0 (\(!\)) if \( \cos \theta = 0 \).
Total derivative of $f$ at $\hat{a}$:

This is a linear transformation $T_{\hat{a}} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$E(\hat{a}; h) := \frac{f(\hat{a} + h) - f(\hat{a}) - T_{\hat{a}}(h)}{||h||} \rightarrow 0 \text{ as } ||h|| \rightarrow 0.$$ 

If it exists, then $f$ is differentiable at $\hat{a}$.

**Theorem:** If $f$ is differentiable at $\hat{a}$, then

(a) the partial derivatives exist and $T_{\hat{a}}(\hat{y}) = \sum_{k=1}^{n} \left( \frac{\partial f}{\partial x_k}(\hat{a}) \right) y_k$

(b) the derivatives w.r.t. $\hat{y}$ exist and are $f'(\hat{a}; \hat{y}) = T_{\hat{a}}(\hat{y})$.

**Proof:**

$$f'(\hat{a}; \hat{y}) = \lim_{h \rightarrow 0} \frac{f(\hat{a} + h\hat{\hat{y}}) - f(\hat{a})}{h} = hT_{\hat{a}}(\hat{y}) \text{ by linearity}$$

$$= \lim_{h \rightarrow 0} \frac{h||h|| E(\hat{a}; h) + T_{\hat{a}}(h\hat{\hat{y}})}{h}$$

$$= \lim_{h \rightarrow 0} \frac{||h|| E(\hat{a}; h)}{h} + T_{\hat{a}}(h\hat{\hat{y}})$$

$$= T_{\hat{a}}(\hat{y}).$$

Since $T_{\hat{a}}$ is linear, any $\hat{y} = \sum y_k \hat{e}_k$, we set

$$T_{\hat{a}}(\hat{y}) = \sum_{k} y_k T_{\hat{a}}(\hat{e}_k) = \sum_{k} y_k f'(\hat{a}; \hat{e}_k) = \sum_{k} y_k \frac{\partial f}{\partial x_k}(\hat{a}). \quad \square$$

**Ex 4:** Reversing the function from Ex. 2 yet again, note that since $f'(\hat{a}; \hat{y}) = \begin{cases} \frac{y_2}{y_1}, & y_1 \neq 0 \\ 0, & y_1 = 0 \end{cases}$ is nonlinear in $\hat{y}$, $f$ is not differentiable at $\hat{a}$ (even though its partial derivatives exist!). //
• **Gradients**: \( \nabla f := (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \) is a function from \( \mathbb{R}^n \to \mathbb{R} \), i.e. a "vector field". Note that

\[
T_\alpha (\cdot) = \sum_{k=1}^n \frac{\partial f}{\partial x_k} \cdot \hat{e}_k = (\nabla f(\alpha)) \cdot \hat{y} \quad \text{or} \quad (\nabla f(\alpha)) \hat{y},
\]

which is consistent with a linear transform from \( \mathbb{R}^n \to \mathbb{R} \) being given by a \( 1 \times n \) matrix.

• **Gradient & directional derivative**: give a unit vector \( \hat{u} \),

\[
(D_{\hat{u}} f) (\alpha) = f'(\alpha; \hat{u}) = T_\alpha (\hat{u}) = \nabla f(\alpha) \cdot \hat{u} = \|\nabla f(\alpha)\| \cos \theta
\]

when \( \theta \) is the angle between \( \nabla f(\alpha) \) and \( \hat{u} \).

• **Differentiability → Continuity** (of course, converse is false)

Proof: \( 0 \leq \left| f(\alpha + h) - f(\alpha) \right| = \left| \nabla f(\alpha).h + \|h\| E(\alpha; h) \right| \)

by triangle inequality

\[
\leq \|\nabla f(\alpha)\| \|h\| + \|h\| \|E(\alpha; h)\| \rightarrow 0
\]

as \( \|h\| \to 0 \)

• **Theorem**: If \( \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \) exist in a ball about \( \alpha \)

and are continuous at \( \alpha \), then \( f \) is differentiable at \( \alpha \).

Proof: (WLOG \( \epsilon = 0 \)) We need to show that

\[
(\xi) \quad \frac{f(\alpha + h) - f(\alpha) - \nabla f(\alpha) \cdot h}{\|h\|} \rightarrow 0
\]

\( \)as \( \|h\| \to 0 \).
So \( C^h \) becomes
\[
\prod \left\{ \sum_{v} w_v \left( \frac{\partial f}{\partial x_v} (\xi_v) - \frac{\partial f}{\partial x_v} (\delta) \right) \right\},
\]
which goes to zero by continuity of the \( \partial f/\partial x_v \)'s — since \( h \to 0 \) forces \( \xi_v \to \delta \).

For the function of Examples 2-4, while its partial derivatives exist, they clearly can't be continuous at \( \delta \). Which makes sense, b/c the function itself isn't continuous!