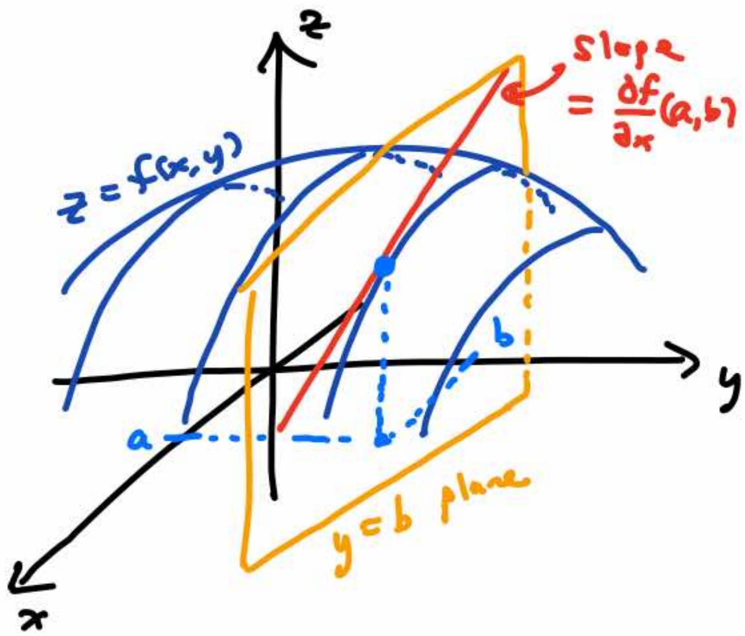
Lecture 26: Partial Derivatives

Given a function $f: S \to \mathbb{R}$ defined on a set $S \subseteq \mathbb{R}^n$, the idea of partial derivatives is to hold all variables but one — say, x_k — constant, and differentiate with respect to x_k to get "$\frac{\partial f}{\partial x_k}$" or "$D_k f$". If $n=2$, we often write (x,y) instead of $\vec{x}=(x_1,x_2)$ and the geometric idea is to slice the graph $z=f(x,y)$ by the plane $y=b$, then compute the slope of $z=f(x,b)$ at $x=a$:



$$\frac{\partial f}{\partial x}(a,b) := \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}.$$

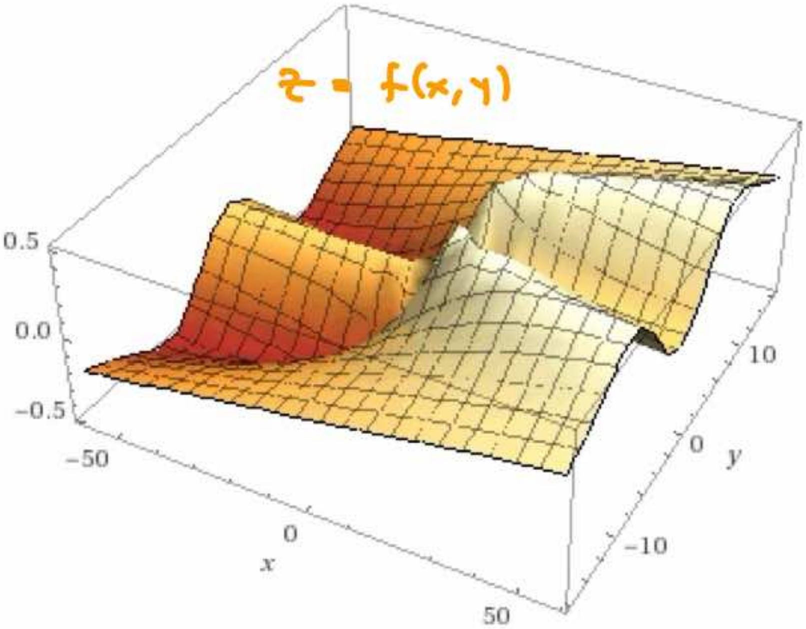
More generally, $\frac{\partial f}{\partial x} := \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ & $\frac{\partial f}{\partial y} := \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$.

Ex 1/ $f(x,y) = \sin\left(\frac{x}{1+y}\right) \rightsquigarrow \frac{\partial f}{\partial x} = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)$ and

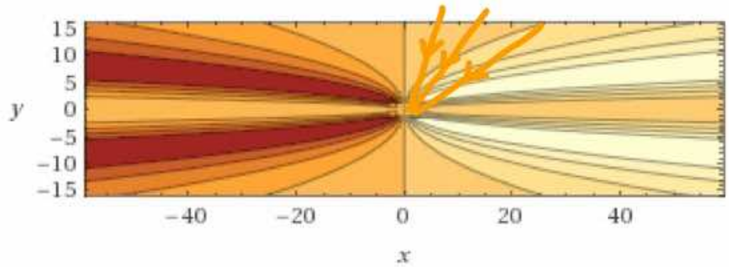
$$\frac{\partial f}{\partial y} = \left(\frac{\partial}{\partial y}\frac{x}{1+y}\right)\cos\left(\frac{x}{1+y}\right) = \frac{-x}{(1+y)^2}\cos\left(\frac{x}{1+y}\right). \quad //$$

Ex 2/ $f(x,y) := \begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0). \end{cases}$ We showed last time that this was not continuous at $(0,0)$ (because limits along paths $x=y^2$ and $y=mx$ disagree). But the partials do exist: restricting to $y=0$ and differentiating w.r.t. x gives $\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx} f(x,0) \Big|_{x=0} = \frac{d}{dx} 0 \Big|_{x=0} = 0$;

and restricting to $x=0$ etc. gives $\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy} f(0,y) \Big|_{y=0} = \frac{d}{dy} 0 = 0$.



The graph



Contour plot

(level curves)

— I have drawn a few of the linear paths into (0,0) that gave limits = 0.

There are essentially 3 cases to consider:

- partials exist and are continuous at (a,b) $\Longrightarrow$ $z = f(x,y)$ is well-approximated by a "tangent plane" at $(a,b,f(a,b))$
- partials don't exist at (a,b) $\Longrightarrow$ something like  or 
- partials exist but are not continuous at (a,b)
  — the weird in-between case we find ourselves in above (Ex. 2)

We don't consider $f(x,y)$ "differentiable" at (a,b) in the 2nd or 3rd cases.

Having done the overview, let's turn to definitions, again in the general case of $\mathbb{R}^n \supseteq S \xrightarrow{f} \mathbb{R}$, with $\vec{a} \in \operatorname{int}(S)$.

Derivative with respect to a vector $\vec{y} \in \mathbb{R}^n$:

$$\boxed{f'(\vec{a};\vec{y}) := \lim_{h\to 0} \frac{f(\vec{a}+h\vec{y}) - f(\vec{a})}{h}}$$

- $f'(\vec{a};\vec{0}) = 0$
- $f(\vec{x}) = \vec{l}\cdot\vec{x}$ linear $\Rightarrow f'(\vec{a};\vec{y}) = \lim_{h\to 0} \frac{\overbrace{f(\vec{a}+h\vec{y})}^{= f(\vec{a}) + h f(\vec{y})} - f(\vec{a})}{h} = f(\vec{y})$.
- $g(t) := f(\vec{a}+t\vec{y}) \Longrightarrow g'(t) = f'(\vec{a}+t\vec{y};\vec{y})$

  (e.g. if $f(\vec{x}) = \|\vec{x}\|^2 = \vec{x}\cdot\vec{x}$, $f'(\vec{a};\vec{y}) = g'(0) = \frac{d}{dt}(\vec{a}+t\vec{y})\cdot(\vec{a}+t\vec{y})\Big|_{t=0}$
  $= 2\vec{a}\cdot\vec{y}$)
- Mean-value theorem for $g(t)$ on $[0,1] \Longrightarrow$

  $$f(\vec{a}+\vec{y}) - f(\vec{a}) = f'(\vec{a}+t_0\vec{y};\vec{y}) \text{ for some } t_0 \in (0,1).$$
- This is called the directional derivative $D_{\vec{u}} f$ if $\vec{y} = \vec{u}$ is a unit vector
- partial derivatives $\frac{\partial f}{\partial x_k}(\vec{a}) = D_k f(\vec{a}) := D_{\vec{e}_k} f(\vec{a}) = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{e}_k) - f(\vec{a})}{h}$

  are a special case of directional derivatives

Ex 3/ f as in Ex. 2, $\vec{y} = \overbrace{(\alpha,\beta)}^{\text{not both } 0}$, $\vec{a} = (0,0)$

$$g(t) = f(\alpha t, \beta t) = \frac{\alpha\beta^2 t}{\alpha^2 + \beta^4 t^2}$$

$$\Rightarrow f'(\vec{0};\vec{y}) = g'(0) = \frac{(\alpha^2+\beta^4t^2)\alpha\beta^2 - \alpha\beta^2 t(2\beta^4 t)}{(\alpha^2+\beta^4t^2)^2}\Bigg|_{t=0} = \frac{\beta^2}{\alpha} \text{ if } \alpha \neq 0$$

(or 0 if $\alpha = 0$, since then $g(t) \equiv 0$)

$\Longrightarrow$ all the $\underbrace{\text{directional derivatives}}_{\|\vec{y}\|=1 \Rightarrow \alpha^2+\beta^2=1\text{, or rather } \alpha=\cos\theta,\ \beta=\sin\theta}$ exist at $(0,0)$

with value $\frac{\sin^2\theta}{\cos\theta}$ if $\cos\theta \neq 0$, and 0 (!) if $\cos\theta = 0$. //

Total derivative of f at $\vec{a}$:

This is a linear transformation $T_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}$ such that

$$E(\vec{a};\vec{h}) := \frac{f(\vec{a}+\vec{h}) - f(\vec{a}) - T_{\vec{a}}(\vec{h})}{\|\vec{h}\|} \longrightarrow 0 \quad \text{as} \quad \|\vec{h}\| \to 0.$$

If it exists, then f is differentiable at $\vec{a}$.

- **Theorem**: If f is differentiable at $\vec{a}$, then
  (a) the partial derivatives exist and $T_{\vec{a}}(\vec{y}) = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}(\vec{a})\right) y_k$
  (b) the derivatives w.r.t. $\vec{y}$ exist and are $f'(\vec{a};\vec{y}) = T_{\vec{a}}(\vec{y})$.

Proof: $f'(\vec{a};\vec{y}) = \lim_{h\to 0} \frac{f(\vec{a}+h\vec{y}) - f(\vec{a})}{h}$

$\vec{h} = h\vec{y} \rightarrow$

$$= \lim_{h\to 0} \frac{|h|\,\|\vec{y}\|\, E(\vec{a};\vec{h}) + \overbrace{T_{\vec{a}}(h\vec{y})}^{= h T_{\vec{a}}(\vec{y}) \text{ by linearity}}}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}\|\vec{y}\| \underbrace{E(\vec{a};\vec{h})}_{\to 0} + T_{\vec{a}}(\vec{y})$$

$$= T_{\vec{a}}(\vec{y}).$$

Since $T_{\vec{a}}$ is linear, any $\vec{y} = \sum y_k \vec{e}_k$, we set

$$T_{\vec{a}}(\vec{y}) = \sum_k y_k T_{\vec{a}}(\vec{e}_k) = \sum_k y_k f'(\vec{a};\vec{e}_k) = \sum_k y_k \frac{\partial f}{\partial x_k}(\vec{a}). \qquad \square$$

Ex 4/ Revisiting the function from Ex. 2 yet again,

notice that since $f'(\vec{0};\vec{y}) = \begin{cases} y_2^2/y_1 & , \; y_1 \neq 0 \\ 0 & , \; y_1 = 0 \end{cases}$

is nonlinear in $\vec{y}$, f is not differentiable at $\vec{0}$ (even though its partial derivatives exist!). //

- <u>Gradients</u>: $\vec{\nabla} f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ is a function from $\mathbb{R}^n \supset S \longrightarrow \mathbb{R}^n$, i.e. a "vector field". Notice that

$$T_{\vec{a}}(\vec{y}) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\vec{a})\, y_k = (\vec{\nabla} f(\vec{a})) \cdot \vec{y} \quad \text{or} \quad \underbrace{{}^t(\vec{\nabla} f(\vec{a}))}\vec{y},$$

which is consistent with a linear transformation from $\mathbb{R}^n \to \mathbb{R}$ being given by a $1 \times n$ matrix.

- <u>Gradients & directional derivatives</u>: given a unit vector $\vec{u}$,

$$(D_{\vec{u}} f)(\vec{a}) = f'(\vec{a};\vec{u}) = T_{\vec{a}}(\vec{u}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u} = \|\vec{\nabla} f(\vec{a})\| \cos\theta$$

where θ is the angle between $\vec{\nabla} f(\vec{a})$ and $\vec{u}$.

- <u>Differentiability $\Longrightarrow$ Continuity</u> (of course, converse is <u>false</u>)

<u>Proof</u>: $0 \le |f(\vec{a}+\vec{h}) - f(\vec{a})| = |\vec{\nabla} f(\vec{a}) \cdot \vec{h} + \|\vec{h}\| E(\vec{a};\vec{h})|$

$$\overset{\text{by } \Delta \text{ inequality + Cauchy-Schwarz}}{\le} \|\vec{\nabla} f(\vec{a})\| \|\vec{h}\| + \|\vec{h}\| |E(\vec{a},\vec{h})| \underset{\text{as } \|\vec{h}\| \to 0}{\longrightarrow} 0 \qquad \square$$

- <u>Theorem</u>: If $\partial f/\partial x_1, \dots, \partial f/\partial x_n$ exist in a ball about $\vec{a}$ and are continuous at $\vec{a}$, then f is differentiable at $\vec{a}$.

<u>Proof</u>: (WLOG $\vec{a} = \vec{0}$) We need to show that

$$(*) \quad \frac{f(\vec{h}) - f(\vec{0}) - \vec{\nabla} f(\vec{0}) \cdot \vec{h}}{\|\vec{h}\|} \longrightarrow 0 \quad \text{as } \|\vec{h}\| \to 0.$$

Write $\vec{h} = h\vec{u}$, $\vec{u}$ = unit vector, $\vec{v}_k = \sum_{j=1}^{k} u_j \vec{e}_j$

$$\Rightarrow f(h\vec{u}) - f(\vec{0}) = \sum_{k=1}^{n} \{ f(\underbrace{h\vec{v}_k}_{h\vec{v}_{k-1} + hu_k\vec{e}_k}) - f(h\vec{v}_{k-1}) \} \overset{\text{MVT from above}}{=} \sum_{k=1}^{n} h u_k \frac{\partial f}{\partial x_k}(\vec{c}_k)$$

on segment connecting $h\vec{v}_{k-1}$ & $h\vec{v}_k$

$$= h \sum_{k=1}^{n} u_k \frac{\partial f}{\partial x_k}(\vec{c}_k)$$

So $(*)$ becomes $$\frac{\cancel{h}\left\{ \sum_k u_k \left(\frac{\partial f}{\partial x_k}(\vec{c}_k) - \frac{\partial f}{\partial x_k}(\vec{0}) \right) \right\}}{\cancel{h}\,\cancelto{1}{\|\vec{u}\|}},$$

which goes to zero by continuity of the $\partial f/\partial x_k$'s — since $h \to 0$ forces $\vec{c}_k \to \vec{0}$. $\square$

For the function of Examples 2-4, while its partial derivatives exist, they clearly can't be *continuous* at $\vec{0}$. Which makes sense, b/c the function itself isn't continuous!