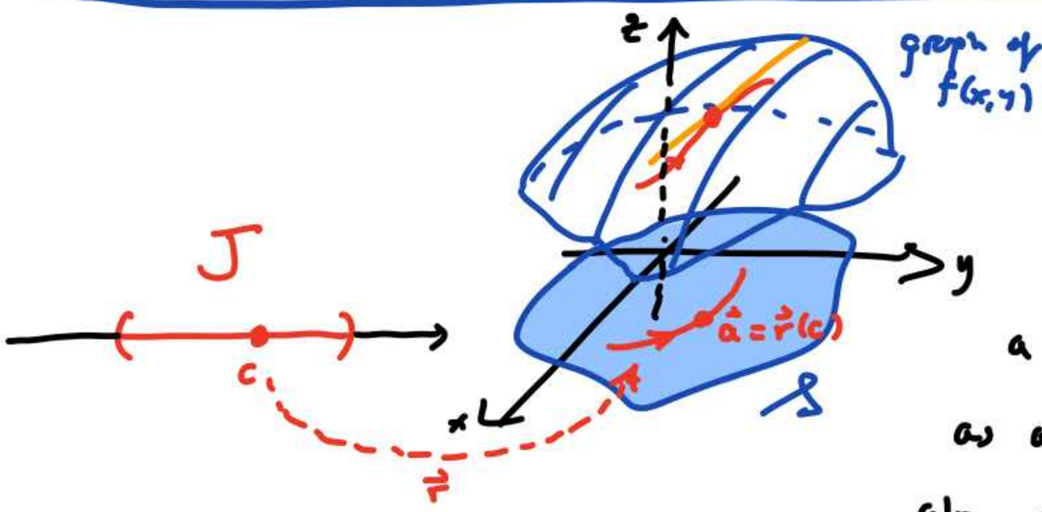
Lecture 27: Chain rule; Jacobians



We want to measure the rate of change of a function f defined on S as a particle moves through it along the parametric curve $\vec r(t)$.

That is, we want to differentiate the composite function g assuming that f is differentiable at $\vec a$ and $\vec r$ is differentiable at c:

$$\begin{tikzcd} c \arrow[r, mapsto] & \vec r(c) =: \vec a \\ J \arrow[r, "\vec r"] \arrow[rr, bend right, "g := f\circ \vec r"'] & S \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$c \in J \subseteq \mathbb{R}$, $\quad \vec a \in S \subseteq \mathbb{R}^n$

coords.: t $\qquad x_1, \dots, x_n$

writing $\vec y = \vec r(c+h) - \vec r(c)$ (so $\vec r(c+h) = \vec a + \vec y$)

$$g'(c) = \lim_{h\to 0} \frac{g(c+h) - g(c)}{h} = \lim_{h\to 0} \frac{f(\vec a + \vec y) - f(\vec a)}{h}$$

$$= \lim_{h\to 0} \frac{\vec\nabla f(\vec a)\cdot \vec y + \|\vec y\|\, E(\vec a; \vec y)}{h} \qquad \text{(by Lecture 26)}$$

$$= \vec\nabla f(\vec a)\cdot \underbrace{\left(\lim_{h\to 0} \frac{\vec r(c+h) - \vec r(c)}{h}\right)}_{\vec r\,'(c)} + \underbrace{\left\|\lim_{h\to 0}\frac{\vec r(c+h)-\vec r(c)}{h}\right\|}_{\vec r\,'(c)} \left(\lim_{h\to 0} \overset{\pm 1}{\frac{|h|}{h}} \underbrace{E(\vec a; \vec y)}_{\to 0}\right)$$

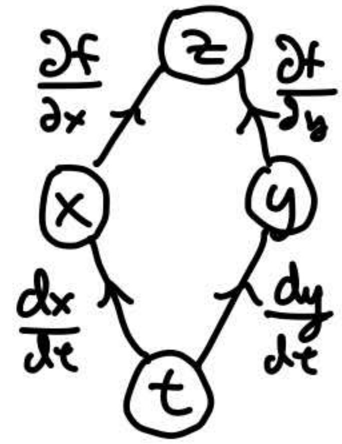
$$= \vec\nabla f(\vec a)\cdot \vec r\,'(c) \ \left(= f'(\vec a; \vec r\,'(c)) \text{ see Lecture 26}\right) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(\vec a)\, \frac{dx_k}{dt}(c)$$

So in the above picture,

i.e. for $n=2$ and $(x,y) = (x_1, x_2)$

this reads $\overset{g'(t)}{\dfrac{dz}{dt}} = \overset{\partial f/\partial x}{\dfrac{\partial z}{\partial x}} \dfrac{dx}{dt} + \overset{\partial f/\partial y}{\dfrac{\partial z}{\partial y}} \dfrac{dy}{dt}$,

reflecting the "dependency diagram":



Ex 1/ Find the directional derivative of $f(x,y) = x^2 - 3xy$ along $y = x^2 - x + 2$ at $(1,2)$.

Parametrize this by $\vec{r}(t) = (x, x^2 - x + 2)$, let $c = 1 \to \vec{a} = \vec{r}(c) = (1,2)$.

$\vec{\nabla} f = (2x - 3y, -3x)$, $\vec{r}\,'(t) = (1, 2x-1)$. For directional derivative though, we need to take $\vec{\nabla} f(1,2) \cdot \underbrace{\dfrac{\vec{r}\,'(1)}{\|\vec{r}\,'(1)\|}}_{\text{unit vector}} = (-4,-3) \cdot \dfrac{(1,1)}{\sqrt{2}}$

$= -7/\sqrt{2}$. //

Ex 2/

Lunch heating in microwave.

At $t=0$, $\begin{cases} r = 10 \text{ cm} \\ h = 2 \text{ cm} \end{cases}$, $\begin{cases} dr/dt = 0.2 \text{ cm/hr} \\ dh/dt = 0.5 \text{ cm/hr} \end{cases}$

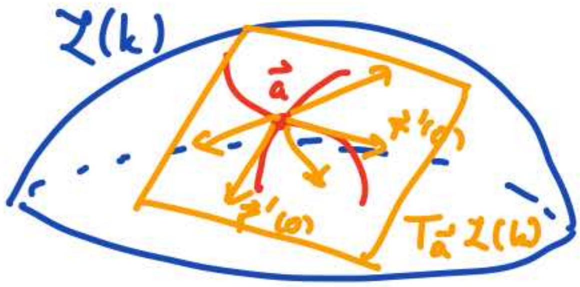
How fast (at that instant) is the volume increasing?

$V = \pi r^2 h \implies \dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt}$, $\dfrac{\partial V}{\partial r} = 2\pi r h$ & $\dfrac{\partial V}{\partial h} = \pi r^2$

At $t=0$, $\dfrac{dV}{dt}(0) = 2\pi(10)(2)\cdot 0.2 + \pi(10)^2 \cdot 0.5 = 8\pi + 50\pi = 58\pi \ \frac{\text{cm}^3}{\text{hr.}}$ //

<u>Definition</u>: Let $\mathcal{L}(k) := \{\vec{x} \in \mathcal{S} \mid f(\vec{x}) = k\}$ be a level set of f, and $\vec{a} \in \mathcal{L}(k)$ a point at which f is differentiable (with nonzero $\vec{\nabla} f(\vec{a})$). The tangent plane $T_{\vec{a}}\mathcal{L}(k)$ is the set $\{\vec{a} + \vec{r}\,'(0) \mid \vec{r} : (-\epsilon, \epsilon) \to \mathcal{L}(k) \text{ differentiable curve with } \vec{r}(0) = \vec{a}\}$.

(Here "plane" is meant as a catch-all term — this has dimension $n-1$.)



A vector is <u>perpendicular</u> to $\mathcal{L}(k)$ at $\vec{a}$ $\iff$ it is perpendicular to $T_{\vec{a}}\mathcal{L}(k)$.

<u>Theorem</u>: $\vec{\nabla} f(\vec{a})$ is perpendicular to $T_{\vec{a}}\mathcal{L}(k)$. Hence the equation of the tangent plane is $\vec{\nabla} f(\vec{a}) \cdot (\vec{x}-\vec{a}) = 0$, and the directional derivative $D_{\hat{u}} f(\vec{a})$ is maximized by taking $\hat{u} = \dfrac{\vec{\nabla} f(\vec{a})}{\|\vec{\nabla} f(\vec{a})\|}$.

<u>Proof</u>: We have (for any curve as above) $g(t) := f(\vec{r}(t)) \equiv k$.

So $0 = g'(t) \Rightarrow 0 = g'(0) = \vec{\nabla} f(\vec{r}(0)) \cdot \vec{r}\,'(0)$
$$= \vec{\nabla} f(\vec{a}) \cdot \vec{r}\,'(0).$$

The difference of any 2 points in $T_{\vec{a}}\mathcal{L}(k)$ is of the form $(\vec{a}+\vec{r}_1{}'(0)) - (\vec{a}+\vec{r}_2{}'(0)) = \vec{r}_1{}'(0) - \vec{r}_2{}'(0)$, which is $\perp$ to $\vec{\nabla} f(\vec{a})$.

The equation is the one from last semester for planes with a normal vector.

The directional derivative is $(D_{\hat{u}} f)(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \hat{u} = \|\vec{\nabla} f(\vec{a})\| (\cos\theta)$, θ = angle between $\vec{\nabla} f(\vec{a})$ & $\hat{u}$. $\square$

So the gradient $\vec{\nabla} f$ points in the direction of fastest increase, which is $\perp$ to the level sets.

Ex 3/ Let $f(x,y) = x^2 + 4y^2$. Find the tangent line to the level set $f(x,y) = 8$ at $(2,1)$.

$\nabla f = (2x, 8y) \Rightarrow \nabla f(2,1) = (4,8) \Rightarrow$ line is $(4,8)\cdot(x-2, y-1) = 0$

i.e. $0 = 4x - 8 + 8y - 8 \Rightarrow x + 2y = 4$. //

Derivatives of vector fields

$$\begin{array}{ccc} S & \xrightarrow{F} & \mathbb{R}^m \\ \cap & & \\ \mathbb{R}^n & & \end{array}$$

Write $F = (f_1, \ldots, f_m) = \sum f_k \vec{e}_k$.

Continuity / differentiability of F can be <u>defined</u> to mean that of $f_1, \ldots, f_m$. The components of

$$(\star) \quad F'(\vec{a};\vec{y}) := \lim_{h\to 0} \frac{F(\vec{a}+h\vec{y}) - F(\vec{a})}{h}$$

are just the $f_k'(\vec{a};\vec{y})$; while the choice of $T_{\vec{a}}(\vec{y})$ making

$$E(\vec{a};\vec{h}) := \frac{F(\vec{a}+\vec{h}) - \left(F(\vec{a}) + T_{\vec{a}}(\vec{h})\right)}{\|\vec{h}\|} \longrightarrow 0$$

(with $\vec{h}\to\vec{0}$)

agrees with $(\star)$ by exactly the same proof as when $m=1$.

So

$$T_{\vec{a}}(\vec{y}) = F'(\vec{a};\vec{y}) = \begin{pmatrix} f_1'(\vec{a};\vec{y}) \\ \vdots \\ f_m'(\vec{a};\vec{y}) \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n \frac{\partial f_1}{\partial x_j}(\vec{a})\, y_j \\ \vdots \\ \sum_{j=1}^n \frac{\partial f_m}{\partial x_j}(\vec{a})\, y_j \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} =: \underbrace{J_F(\vec{a})}\, \vec{y}$$

this is the Jacobian matrix of f at $\vec{a}$ (Apostol writes $DF(\vec{a})$)

$$\left[\frac{\partial f_i}{\partial x_j}(\vec{a})\right]$$

This matrix gives the linear transformation locally approximating $F(\vec{a}+\vec{y}) - F(\vec{a})$.