Lecture 29: Partial Differential Equations

We'll explore these in a sequence of four examples, all for a function \( f \) of 2 variables.

1. \[ a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} = 0, \quad a, b \in \mathbb{R} \]

Begin with a change of coordinates, to \( \begin{cases} u = bx - ay \\ v = ax + by \end{cases} \). Writing \( f(x, y) = g(u(x, y), v(x, y)) \) and applying the chain rule, the equation becomes

\[
0 = a \left( \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial x} \right) + b \left( \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial y} \right) = (ab - ba) \frac{\partial g}{\partial u} + (a^2 + b^2) \frac{\partial g}{\partial v}
\]

\[ \Rightarrow 0 = \frac{\partial g}{\partial v} \Rightarrow g \text{ is constant in } v \Rightarrow g(u, v) = G(u) \]

\[ \Rightarrow f(x, y) = G(u(x, y)) = G(bx - ay). \]

Note that, in the absence of additional constraints, the solution space is 2-dimensional.

In the remaining examples I take \( f \) to be a function of \( x \) (space) and \( t \) (time). Additional constraints typically appear in the form of "if starts at \( t = 0 \) with \( f(x, 0) = \text{given } F(x) \)". In the next example you should imagine this as the starting position of a wave.
Wave equation: \( \frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \)

**Intuition:**
- Concavity of a string induces upward force then acceleration

Remark as

\[
0 = \left( \frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} \right) f = \left( \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} \right) f = \left( \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t} \frac{\partial^2 f}{\partial t^2} \right) f
\]

By \( \textcircled{1} \), \( u(x,t) = \varphi(x+ct) \). Let \( v(x,t) = \frac{1}{2c} \Phi(x+ct) \), where \( \Phi \) is any antiderivative of \( \varphi \). Then

\[
\begin{cases}
\frac{\partial v}{\partial x} = \frac{1}{2c} \Phi'(x+ct) = \frac{1}{2c} \varphi(x+ct) = \frac{u}{2c} \\
\frac{\partial v}{\partial t} = \frac{1}{2c} c \Phi'(x+ct) = \frac{1}{2} \varphi(x+ct) = \frac{u}{2}
\end{cases} \quad \Rightarrow \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) v = u
\]

\[
\Rightarrow \left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) (f - v) = 0 \quad \Rightarrow \quad f - v = \Psi(x - ct)
\]

\[
\Rightarrow \quad f(x,t) = \frac{1}{2c} \Phi(x+ct) + \Psi(x-ct) \quad \text{(4)}
\]

Initial conditions give

\[
\begin{cases}
F(x) = f(x,0) = \frac{1}{2c} \Phi(x) + \Psi(x) \Rightarrow F' = \frac{1}{2c} \Phi' + \Psi' \\
G(x) = \frac{\partial F}{\partial x}(x,0) = \frac{1}{2c} \Phi'(x) - c \Psi'(x)
\end{cases}
\]

Solve

\[
\begin{cases}
\Phi'(x) = c F(x) + G(x) \\
\Psi'(x) = \frac{1}{2} F'(x) - \frac{1}{2c} G(x)
\end{cases} \quad \Rightarrow \begin{cases}
\Phi(y) = c F(y) + \int_0^y G(u) \, du + c
\\
\Psi(y) = \frac{1}{2} F(y) - \frac{1}{2c} \int_0^y G(u) \, du
\end{cases}
\]

Plug into (4)

\[
\Rightarrow \quad f(x,t) = \frac{F(x+ct) + F(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(u) \, du
\]

\( \text{5 shows initial wave spreading out to left & right at speed } c \).

For instance, if \( F(x) = \cos(kx) \) and \( G(x) = 0 \), then

\[
f(x,t) = \frac{\cos(k(x+ct)) + \cos(k(x-ct))}{2} = \cos(kx) \cos(kct)
\]

oscillates (faster for shorter waves).
\[ \frac{\partial f}{\partial t} = c^2 \frac{\partial^2 f}{\partial x^2} \]

with initial condition \[ f(x,0) = F(x) \]

**INTUITION:**

Concavity of temperature distribution \( \leftrightarrow \) rate of increase of temperature (not acceleration of increase)

It's not so easy to solve this as we did (2). Instead we take inspiration from the fact that \( \cos(kx) \) is an eigenfunction of \( \frac{\partial^2}{\partial x^2} \) (see the end of (2)). So we should be able to get a solution by multiplying \( F(x) = \cos(kx) \) by an eigenfunction \( G(t) \) for \( \frac{\partial}{\partial t} \); that is, with \( f(x,t) = G(t) \cos(kx) \), the heat equation gives

\[ G'(t) \cos(kx) = -c^2 k^2 G(t) \cos(kx) \]

\[ \Rightarrow \begin{cases} G'(t) = -c^2 k^2 G(t) \Rightarrow G(t) = e^{-c^2 k^2 t} \\ G(0) = 1 \end{cases} \]

\[ f(x,t) = e^{-c^2 k^2 t} \cos(kx) \]

More generally, suppose we have any \( F(x) \). Then the idea is to write it as a “linear combination” of different frequencies, then multiply each frequency by \( e^{-c^2 k^2 t} \) exponential decay factor. For instance, if \( F \) is periodic with period \( 2\pi \) \( (F(x+2\pi) = F(x)) \), you can try to write

\[ F(x) = \sum_{k=1}^{\infty} (a_k \cos(kx) + b_k \sin(kx)) + c \]

and then

\[ f(x,t) = \sum_{k=1}^{\infty} e^{-c^2 k^2 t} (a_k \cos(kx) + b_k \sin(kx)) + c \]

(This is a taste of what Fourier theory was invented to do.)
Writing \( \tilde{y}(x,t) = (y_1(x,t), y_2(x,t)) \), consider a system

\[
\frac{\partial \tilde{y}}{\partial t} = A \frac{\partial \tilde{y}}{\partial x}
\]

—that is:

\[
\begin{align*}
\frac{\partial y_1}{\partial t} &= a_{11} \frac{\partial y_1}{\partial x} + a_{12} \frac{\partial y_2}{\partial x} \\
\frac{\partial y_2}{\partial t} &= a_{21} \frac{\partial y_1}{\partial x} + a_{22} \frac{\partial y_2}{\partial x}
\end{align*}
\]

(We can also specify \( \tilde{y}(x,0) \).)

If \( A = SDS^{-1} \) is diagonalizable, we can “decouple” the system: set \( \tilde{z} = S^{-1} \tilde{y} \), so that

\[
\frac{\partial \tilde{z}}{\partial t} = D \frac{\partial \tilde{z}}{\partial x}
\]

with

\[
\begin{align*}
\frac{\partial z_1}{\partial t} &= \lambda_1 \frac{\partial z_1}{\partial x} \\
\frac{\partial z_2}{\partial t} &= \lambda_2 \frac{\partial z_2}{\partial x}
\end{align*}
\]

(\( S \) if \( \tilde{y}(x,0) = (\cos(x) \ 0) \) and \( A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix} \),

then \( S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \), \( D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \), \( S^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \) \( \Rightarrow \)

\[
\tilde{z}(x,0) = S^{-1} \begin{pmatrix} \cos(x) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(x) \\ -\cos(x) \end{pmatrix} \Rightarrow \tilde{z}(x,t) = \begin{pmatrix} \cos(x+\lambda_1 t) \\ -\cos(x+\lambda_2 t) \end{pmatrix}
\]

\( \Rightarrow \tilde{y}(x,t) = S \tilde{z}(x,t) = \begin{pmatrix} 2\cos(x+\lambda_1 t) - \cos(x+\lambda_2 t) \\ \cos(x+\lambda_1 t) - \cos(x+\lambda_2 t) \end{pmatrix} \begin{pmatrix} y_1(x,t) \\ y_2(x,t) \end{pmatrix} \) gives the solution.

Check:

\[
\frac{\partial y_1}{\partial t} = 2 \sin(x+\lambda_1 t) + 2 \sin(x+\lambda_2 t) = -4 (-2 \sin(x-t) + \sin(x+2t)) + 6 (-\sin(x-t) + \sin(x+2t))
\]

\[
= -4 \frac{\partial y_1}{\partial x} + 6 \frac{\partial y_2}{\partial x} \ . \ \checkmark
\]