Lecture 29: Partial Differential Equations

We'll explore these in a sequence of four examples, all for a function f of 2 variables.

① $a\dfrac{\partial f}{\partial x} + b\dfrac{\partial f}{\partial y} = 0$, $\quad a,b \in \mathbb{R}$

Begin with a change of coordinates, to $\begin{cases} u = bx - ay \\ v = ax + by \end{cases}$.

Writing $f(x,y) = g(u(x,y), v(x,y))$ and applying the chain rule, the equation becomes

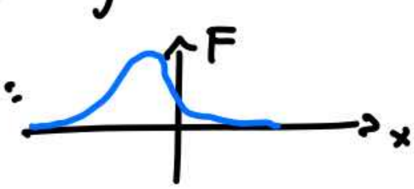
$$0 = a\left(\frac{\partial g}{\partial u}\underbrace{\frac{\partial u}{\partial x}}_{b} + \frac{\partial g}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{a}\right) + b\left(\frac{\partial g}{\partial u}\underbrace{\frac{\partial u}{\partial y}}_{-a} + \frac{\partial g}{\partial v}\underbrace{\frac{\partial v}{\partial y}}_{b}\right) = \underbrace{\cancel{(ab-ba)}}_{0}\frac{\partial g}{\partial u} + (a^2+b^2)\frac{\partial g}{\partial v}$$

$\Rightarrow 0 = \dfrac{\partial g}{\partial v} \Rightarrow g$ is constant in $v \Rightarrow g(u,v) = G(u)$

$\Rightarrow f(x,y) = G(u(x,y)) = G(bx - ay)$.

Note that, in the absence of additional constraints, the solution space is ∞-dimensional.

In the remaining examples I take f to be a function of x (space) and t (time). Additional constraints typically appear in the form of "f starts at $t=0$ with $f(x,0) = $ given $F(x)$". In the next example you should imagine this as the starting position of a wave:



(2) Wave equation : $\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2}$

$$\begin{cases} \text{w/ initial conditions} \\ f(x,0) := F(x) \quad \text{starting position} \\ \dfrac{\partial f}{\partial t}(x,0) := G(x) \quad \text{starting velocity} \end{cases}$$

INTUITION :

concavity of a string induces upward force hence acceleration

Rewrite as

$$0 = \left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right) f = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\overbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) f}^{=:\, u(x,t)}$$

By (1), $u(x,t) = \varphi(x+ct)$. Let $v(x,t) := \frac{1}{2c}\Phi(x+ct)$, where Φ is any antiderivative of φ. Then

$$\begin{cases} \partial v/\partial x = \frac{1}{2c}\Phi'(x+ct) = \frac{1}{2c}\varphi(x+ct) = \frac{u}{2c} \\ \partial v/\partial t = \frac{1}{2c}c\,\Phi'(x+ct) = \frac{1}{2}\varphi(x+ct) = \frac{u}{2} \end{cases} \Longrightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) v = u$$

$$\Longrightarrow \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)(f-v) = 0 \underset{(1)}{\Longrightarrow} f - v = \Psi(x-ct)$$

$$\Longrightarrow \boxed{f(x,t) = \frac{1}{2c}\Phi(x+ct) + \Psi(x-ct).} \quad (*)$$

Initial conditions give
$$\begin{cases} F(x) = f(x,0) = \frac{1}{2c}\Phi(x) + \Psi(x) \Rightarrow F' = \frac{1}{2c}\Phi' + \Psi' \\ G(x) = \frac{\partial f}{\partial t}(x,0) = \frac{1}{2}\Phi'(x) - c\Psi'(x) \end{cases}$$

$$\overset{\text{solve}}{\Longrightarrow} \begin{cases} \Phi'(x) = cF'(x) + G(x) \\ \Psi'(x) = \frac{1}{2}F'(x) - \frac{1}{2c}G(x) \end{cases} \rightarrow \begin{cases} \Phi(y) = cF(y) + \int_0^y G(w)\,dw \quad (+C) \\ \Psi(y) = \frac{1}{2}F(y) - \frac{1}{2c}\int_0^y G(w)\,dw \end{cases}$$

$$\underset{(*)}{\overset{\text{plug in to}}{\Longrightarrow}} f(x,t) = \frac{F(x+ct) + F(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(u)\,du$$

↳ shows initial wave "spreading out" to left & right at "speed" c.

For instance, if $F(x) = \cos(kx)$ and $G(x) = 0$, then

$$f(x,t) = \frac{\cos(k(x+ct)) + \cos(k(x-ct))}{2} = \underbrace{\cos(kx)}_{F(x)}\cos(kct)$$

oscillates (faster for shorter waves)

③ Heat equation $\dfrac{\partial f}{\partial t} = c^2 \dfrac{\partial^2 f}{\partial x^2}$ $\quad \left\{ \begin{array}{l} \text{w/ initial condition} \\ f(x,0) = F(x) \end{array} \right.$

INTUITION: concavity of temperature distribution
$\leftrightarrow$ rate of increase of temperature
(not acceleration of increase)

It's not so easy to solve this as we did ②. Instead we take inspiration from the fact that $\cos(kx)$ is an eigenfunction of $\partial^2/\partial x^2$ (from the end of ②). So we should be able to get a solution by multiplying $F(x) = \cos(kx)$ by an eigenfunction $G(t)$ for $\partial/\partial t$: that is, with $f(x,t) = G(t)\cos(kx)$, the heat equation gives $G'(t)\cos(kx) = -c^2k^2 G(t)\cos(kx)$

$$\Longrightarrow \begin{cases} G'(t) = -c^2k^2G(t) \\ G(0) = 1 \end{cases} \Longrightarrow G(t) = e^{-c^2k^2t}$$

$$\Longrightarrow f(x,t) = e^{-c^2k^2t} \overbrace{\cos(kx)}^{F(x)}$$

exponential decay (faster for shorter waves)

More generally, suppose we have any $F(x)$. Then the idea is to write it as a "linear combination" of different frequencies, then multiply each frequency by its exponential decay factor. For instance, if F is periodic with period 2π $\big(F(x+2\pi) = F(x)\big)$, you can try to write

$$F(x) = \sum_{k=1}^{\infty} \big(a_k \cos(kx) + b_k \sin(kx)\big) + c \quad \text{and then}$$

$$f(x,t) = \sum_{k=1}^{\infty} e^{-c^2k^2t}\big(a_k\cos(kx) + b_k\sin(kx)\big) + c.$$

(This is a taste of what Fourier theory was invented to do.)

4) Systems of 1^{st}-order PDEs

Writing $\vec{y}(x,t) = \begin{pmatrix} y_1(x,t) \\ y_2(x,t) \end{pmatrix}$, consider a system

$$\frac{\partial \vec{y}}{\partial t} = A \frac{\partial \vec{y}}{\partial x} \quad \text{— that is:} \quad \begin{cases} \dfrac{\partial y_1}{\partial t} = a_{11} \dfrac{\partial y_1}{\partial x} + a_{12} \dfrac{\partial y_2}{\partial x} \\ \dfrac{\partial y_2}{\partial t} = a_{21} \dfrac{\partial y_1}{\partial x} + a_{22} \dfrac{\partial y_2}{\partial x} \end{cases}$$

(We can also specify $\vec{y}(x,0)$.)

If $A = SDS^{-1}$ is diagonalizable, we can "decouple" the system: set $\vec{z} = S^{-1}\vec{y}$, so that

$$\frac{\partial \vec{z}}{\partial t} = D \frac{\partial \vec{z}}{\partial x} \iff \begin{cases} \dfrac{\partial z_1}{\partial t} = \lambda_1 \dfrac{\partial z_1}{\partial x} \xRightarrow{(1)} z_1(x,t) = \varphi_1(x+\lambda_1 t) \\ \dfrac{\partial z_2}{\partial t} = \lambda_2 \dfrac{\partial z_2}{\partial x} \xRightarrow{(1)} z_2(x,t) = \varphi_2(x+\lambda_2 t) \end{cases}$$

e.g. if $\vec{y}(x,0) = \begin{pmatrix} \cos(x) \\ 0 \end{pmatrix}$ and $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$,

then $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & \\ & 2 \end{pmatrix}$, $S^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow$

$$\vec{z}(x,0) = S^{-1}\begin{pmatrix} \cos(x) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(x) \\ -\cos(x) \end{pmatrix} \begin{matrix} \varphi_1(x) \\ \varphi_2(x) \end{matrix} \Longrightarrow \vec{z}(x,t) = \begin{pmatrix} \cos(x-t) \\ -\cos(x+2t) \end{pmatrix}$$

$$\Rightarrow \vec{y}(x,t) = S\vec{z}(x,t) = \begin{pmatrix} 2\cos(x-t) - \cos(x+2t) \\ \cos(x-t) - \cos(x+2t) \end{pmatrix} \begin{matrix} y_1(x,t) \\ y_2(x,t) \end{matrix}$$

gives the solution.

Check: $\dfrac{\partial y_1}{\partial t} = 2\sin(x-t) + 2\sin(x+2t) = -4\left(-2\sin(x-t) + \sin(x+2t)\right) + 6\left(-\sin(x-t) + \sin(x+2t)\right)$

$= -4\dfrac{\partial y_1}{\partial x} + 6\dfrac{\partial y_2}{\partial x}$. ✓