Lecture 3: Matrix Equations

Multiplying a matrix by a (column) vector: 2 approaches

1. Row-by-column (more computationally efficient)

\[
\begin{pmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  a_1 x + b_1 y + c_1 z \\
  a_2 x + b_2 y + c_2 z
\end{pmatrix}.
\]

Notice that this
\[
= (a_1 x) + (b_1 y) + (c_1 z)
= a_1 x + b_1 y + c_1 z,
\]

which leads to...

2. Linear combinations (more conceptual)

Writing \( A = \begin{pmatrix} \hat{v}_1 & \hat{v}_2 & \ldots & \hat{v}_n \end{pmatrix} \) and \( x = \begin{pmatrix} x_1 \\
  x_2 \\
  \vdots \\
  x_n \end{pmatrix} \),

\[
A x = x_1 \hat{v}_1 + x_2 \hat{v}_2 + \ldots + x_n \hat{v}_n.
\]
As an example of the second approach's utility, we can use it to check the

**Linearity property:** \( A(c \vec{v} + d \vec{w}) = cA\vec{v} + dA\vec{w} \)

By (\( \ast \)),

\[
\text{LHS} = (c u_1 + d w_1) \vec{v}_1 + \cdots + (c u_n + d w_n) \vec{v}_n
\]
\[
= c(u_1 \vec{v}_1 + \cdots + u_n \vec{v}_n) + d(w_1 \vec{v}_1 + \cdots + w_n \vec{v}_n)
\]
\[
= \text{RHS}.
\]

More importantly, we see that

**Statement 1:** \( A\vec{x} = \vec{b} \) has a solution in \( \vec{x} \)

(i.e. is consistent)

is equivalent to

**Statement 2:** \( \vec{b} \in \text{Span}\{\text{columns of } A\} \)

The link is (\( \ast \)): \( \vec{b} = A\vec{x} = c_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n \) says "you can choose \( x_1, \ldots, x_n \) so that \( \vec{b} \) is a linear combination of \( \vec{v}_1, \ldots, \vec{v}_n \)."

So to check Statement 2, you just row-reduce \([A | \vec{b}]\).
Here is another set of equivalent statements:

**Assertion (A):** \( \text{span} \{ \text{columns of } A \} = (\text{all } \neq 0) \mathbb{R}^m \)

\[ \Downarrow \] (clear from above)

**Assertion (B):** \( A \vec{z} = \vec{b} \) is consistent for any \( \vec{b} \)

\[ \Downarrow \] **Why?**

**Assertion (C):** \( \text{rref}(A) \) has no rows of all zeroes.

First,

- \( [A \mid \vec{b}] \sim [\text{rref}(A) \mid \vec{c}] \) for some vector \( \vec{c} \)
  
  (apply the sequence of row operations that puts \( A \) in \text{RREF}, to the augmented matrix)

- we can choose \( \vec{b} \) so that \( \vec{c} \) is any vector
  
  (because row operations are reversible)

\[ (C) \Rightarrow (B) : \] If \( (C) \) holds, then \( [\text{rref}(A) \mid \vec{c}] \)

has a leading ‘1’ in every row in the “\( \text{rref}(A) \)” part, hence is in \text{RREF} itself and = \( \text{rref}[A \mid \vec{b}] \). Since the leading ‘1’s occur to the left of \( \vec{c} \), \( \vec{c} \) is not a pivot column and the system is consistent (regardless of \( \vec{b} \)).

So \( (B) \) holds.
(B) ⇒ (C): If (C) fails, choose $\tilde{c}$ so that $\tilde{c}$ has a nonzero last entry. Since the last row of $\text{ref}(A)$ is all '0's, $\tilde{c}$ is a pivot column (for this choice of $\tilde{b}$), and so (B) fails.

Ex 1: For which $\tilde{b}$ is $\begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix} x = \tilde{b}$ solvable?

(Equivalent question: determine $\text{span}\{\begin{pmatrix} 3 \\ -9 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix}\}$.)

$\begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{3} \\ 9 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

So we must have $b_2 = -3b_1$, i.e. $\tilde{b}$ must be a scalar multiple of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Ex 2: Do the columns of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ span $\mathbb{R}^3$?

(Equivalent question: does $\text{ref}(A)$ have no rows of all 0's?)

Row-reduce:

$A \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -4 & 12 \\ 0 & -8 & -16 & -24 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

Answer: NO.
Ex 3/ Suppose $A = 4 \times 4$ matrix, and $\mathbf{b} \in \mathbb{R}^4$ are such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Must columns of $A$ span $\mathbb{R}^4$?

Consider the augmented matrix $[A \mid \mathbf{b}]$ and its ref, which must have:
- at least one non-pivot column (for existence of a solution)
- no non-pivot columns in the "[ ]" part of the augmented matrix (for uniqueness)

So all columns of $A$ are pivot } \Rightarrow \text{ all rows of ref have a leading '1'}

\[ \begin{align*}
\text{And } A & \quad 4 \times 4 \\
\Rightarrow & \quad \text{no rows of zeroes} \\
\Rightarrow & \quad \text{columns span } \mathbb{R}^4
\end{align*} \]

Notice that if $A$ was instead $5 \times 4$, this argument breaks down and columns need not span $\mathbb{R}^5$ (in fact, they can't).

In lecture 2, I claimed the

**Theorem**: Every matrix $A$ is row-equivalent to a unique RREF matrix.
To show this, we need the following:

(*) If $A$ is row-equivalent to $B$, then the rows of $A$ are linear combinations of rows of $B$ (and vice versa).

This is because

- the row operations—replace, swap, scale—simply replace a given row by a linear combination of rows.
- they are reversible.

To prove the Theorem, proceed in 3 steps:

1. It is enough to show that

   (***) 2 row-equivalent RREF matrices $U$ and $V$ must be the same.

   Why? By the algorithm, $A \sim \text{Rref}(A) =: U$.

   If $A \sim V$ (also RREF), then $U \sim V$.

   (So if (***) is known, we get $U = V$ as desired.)

For the remaining 2 steps (which prove (***) let $U$ and $V$ be 2 row-equivalent RREF matrices:
The pivot columns of $U$ & $V$ are the same.

Write $\vec{u}_1, \vec{u}_2, \ldots$ for rows of $U$ (viewed as vectors), same for $V$.

(A) By (x), $\vec{u}_1 \in \text{span} \{\text{rows of } V\}$. So:
$$\vec{u}_1 = a_1 \vec{v}_1 + a_2 \vec{v}_2 + \ldots \Rightarrow i_1 \geq j_1.$$ Reversing $U$ & $V$ gives $j_1 \geq i_1$. So $i_1 = j_1$ (and $a_1 = 1$).

(B) Next, $\vec{u}_2 = b_1 \vec{v}_1 + b_2 \vec{v}_2 + \ldots \Rightarrow \vec{u}_2$ has $b_1$ in ($j_1$-)th coord.
$$\Rightarrow b_1 = 0$$
$$\Rightarrow \vec{u}_2 = b_2 \vec{v}_2 + b_3 \vec{v}_3 + \ldots$$ (and vice versa).

(C) Striking out the first row of $U$ and $V$, the remaining (still RAEF!) matrices are row equivalent. Go back to (A), find $i_2 = j_2$, and repeat until there's nothing left.

The non-pivot columns of $U$ & $V$ are equal.

(D) We have $\vec{u}_1 = \vec{v}_1 + a_2 \vec{v}_2 + \ldots + a_m \vec{v}_m$ (recall $a_1 = 1$ above)
$$\Rightarrow \vec{u}_1 \text{ has } a_2 \text{ in the } (j_2=)\text{ }i_2\text{th coordinate}$$
$$a_3 \text{ in the } (j_3)\text{ }i_3\text{th coordinate}, \text{ etc}$$
$$\Rightarrow 0 = a_2 = a_3 = \ldots = a_m$$
So \( \hat{v}_1 = \hat{v}_1 \).

(E) Strike out the first row of \( U \& V \), go back to (D),
get \( \hat{v}_2 = \hat{v}_2 \), and so on.