# Lecture 3 : Matrix Equations

<u>Multiplying a matrix by a (column) vector</u> : 2 approaches

① <u>Row-by-column</u> (more computationally efficient)

row vectors

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \underset{\text{definition}}{:=} \begin{pmatrix} a_1 x + b_1 y + c_1 z \\ a_2 x + b_2 y + c_2 z \end{pmatrix}.$$

take "dot products"

Notice that this $= \begin{pmatrix} a_1 x \\ a_2 x \end{pmatrix} + \begin{pmatrix} b_1 y \\ b_2 y \end{pmatrix} + \begin{pmatrix} c_1 z \\ c_2 z \end{pmatrix}$

$$= x \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + y \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} + z \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

which leads to ...

② <u>Linear combinations</u> (more conceptual)

Writing $A = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$,

column vectors

(*) $$\boxed{A\vec{x} = x_1 \vec{v}_1 + x_2 \vec{v}_2 + \cdots + x_n \vec{v}_n}$$

As an example of the second approach's utility, we can use it to check the

**Linearity property**: $A\underbrace{(c\vec{u}+d\vec{w})}_{\begin{pmatrix} cu_1+dw_1 \\ \vdots \\ cu_n+dw_n \end{pmatrix}} = cA\underbrace{\vec{u}}_{\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}} + dA\underbrace{\vec{w}}_{\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}}$

By $(*)$,

$$
\begin{aligned}
\text{LHS} &= (cu_1+dw_1)\vec{v}_1 + \cdots + (cu_n+dw_n)\vec{v}_n \\
&= c(u_1\vec{v}_1+\cdots+u_n\vec{v}_n) + d(w_1\vec{v}_1+\cdots+w_n\vec{v}_n) \\
&= \text{RHS}.
\end{aligned}
$$

More importantly, we see that

Statement 1: $A\vec{x}=\vec{b}$ has a solution in $\vec{x}$ (i.e. is consistent)

$\Updownarrow$ is equivalent to

Statement 2: $\vec{b} \in \operatorname{span}\{\text{columns of } A\}$

The link is $(*)$: $\vec{b} = A\vec{x} \overset{(*)}{=} x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n$ says "you can choose $x_1,\ldots,x_n$ so that $\vec{b}$ is a linear combination of $\vec{v}_1,\ldots,\vec{v}_n$".

So to check Statement 2, you just row-reduce $[A \mid \vec{b}]$.

there is another set of equivalent statements:

<u>Assertion (A)</u>: Span{columns of $A$} = (all of) $\mathbb{R}^m$

$\Updownarrow$ (clear from above)

<u>Assertion (B)</u>: $A\vec{x} = \vec{b}$ is consistent for any $\vec{b}$

$\Updownarrow$ why?

<u>Assertion (C)</u>: rref(A) has no rows of all zeroes.

First,

- $[A \mid \vec{b}] \underset{\text{row-equiv.}}{\sim} [\text{rref}(A) \mid \vec{c}]$ for some vector $\vec{c}$

  (apply the sequence of row operations that puts A in RREF, to the <u>augmented</u> matrix)

- we can choose $\vec{b}$ so that $\vec{c}$ is any vector

  (because row operations are reversible)

<u>$(C) \Longrightarrow (B)$</u>: If (C) holds, then <u>$[\text{rref}(A) \mid \vec{c}]$</u> has a leading '1' in every row in the "rref(A)" part, hence is in RREF itself and <u>$= \text{rref}[A \mid \vec{b}]$</u>. Since the leading '1's occur to the left of $\vec{c}$, $\vec{c}$ is not a pivot column and the system is consistent (regardless of $\vec{b}$). So (B) holds.

$\underline{(B) \Rightarrow (C)}$: If (C) fails, choose $\vec{b}$ so that $\vec{c}$ has a nonzero last entry. Since the last row of rref(A) is all '0's, $\vec{c}$ is a pivot column (for this choice of $\vec{b}$), and so (B) fails.

**Ex 1/** For which $\vec{b}$ is $\begin{pmatrix} 3 & -1 \\ -9 & 3 \end{pmatrix} \vec{x} = \vec{b}$ solvable?

(Equivalent question: determine $\text{span}\left\{ \begin{pmatrix} 3 \\ -9 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right\}$.)

failure of (C)

$$\left[\begin{array}{cc|c} 3 & -1 & b_1 \\ -9 & 3 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & b_1/3 \\ -9 & 3 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{3} & b_1/3 \\ 0 & 0 & b_2 + 3b_1 \end{array}\right]$$

so we must have $b_2 = -3b_1$, i.e. $\vec{b}$ must be a scalar multiple of the vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$. //

failure of (A)

**Ex 2/** Do the columns of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ span $\mathbb{R}^3$?

(Equivalent question: does rref(A) have no rows of "all 0"?)

Row-reduce:

$$A \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -24 \end{pmatrix} \to \underbrace{\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{= \text{rref}(A)}$$

Answer: $\underline{\text{NO}}$. //

Ex 3/ Suppose $A = 4\times 4$ matrix, and $\vec{b} \in \mathbb{R}^4$ are such that $A\vec{x} = \vec{b}$ has a unique solution. Must columns of $A$ span $\mathbb{R}^4$?

Consider the augmented matrix $[A \mid \vec{b}]$ and its rref, which must have

- $\vec{b}$ non-pivot column (for existence of a solution)
- no $\underbrace{\text{non-pivot columns}}_{= \text{free variables}}$ in the "[ ⋯ |" part of the augmented matrix (for uniqueness)

So all columns of $A$ are pivot AND $A$ $4\times 4$ $\Rightarrow$ all rows of rref have a leading '1'

$\Rightarrow$ no rows of zeroes

$\Rightarrow$ columns span $\mathbb{R}^4$.

Notice that if $A$ was instead $5\times 4$, this argument breaks down and columns need not span $\mathbb{R}^5$ (in fact, they can't). //

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In lecture 2, I claimed the

**Theorem**: Every matrix $A$ is row-equivalent to a unique RREF matrix.

To show this, we need the following:

(*) If $A$ is row-equivalent to $B$, then the rows of $A$ are linear combinations of rows of $B$ (and vice versa).

This is because

- the row operations — replace, swap, scale — simply replace a given row by a linear combination of rows.
- they are reversible.

To prove the Theorem, proceed in 3 steps:

1. It is enough to show that

(**) 2 row-equivalent RREF matrices $U$ and $V$ must be the same.

Why? By the algorithm, $A \underset{\text{row-eq.}}{\sim} \text{rref}(A) =: U$.

If $A \underset{\text{row-eq.}}{\sim} V$ (also RREF), then $U \underset{\text{row-eq}}{\sim} V$.

(So if (**) is known, we get $U = V$ as desired.)

For the remaining 2 steps (which prove (**)), let $U$ and $V$ be 2 row-equivalent RREF matrices:

## (2) The pivot columns of $U$ & $V$ are the same.

$\boxed{\text{picture of } U}$

$$\begin{array}{l}
\begin{bmatrix} 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & 0 & \cdots \\ 0 \cdots\cdots & \cdots & \cdots 0 & 1 & * \cdots * & 0 & \cdots \\ 0 \cdots\cdots & \cdots & \cdots & \cdots & \cdots 0 & 1 & \cdots \\ & & \vdots & & & & \end{bmatrix} \\
\text{column: } \quad i_1 \qquad\qquad i_2 \qquad\quad i_3 \ \cdots
\end{array}$$

$\boxed{\text{picture of } V}$

$$\begin{array}{l}
\begin{bmatrix} 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & 0 & \cdots \\ 0 \cdots\cdots & \cdots & \cdots 0 & 1 & * \cdots * & 0 & \cdots \\ 0 \cdots\cdots & \cdots & \cdots & \cdots & \cdots 0 & 1 & \cdots \\ & & \vdots & & & & \end{bmatrix} \\
\text{column: } \quad j_1 \qquad\qquad j_2 \qquad\quad j_3 \ \cdots
\end{array}$$

Write $\vec{u}_1, \vec{u}_2$, etc. for rows of $U$ (viewed as vectors), same for $V$.

(A) By $(*)$, $\vec{u}_1 \in \text{Span}\{\text{rows of } V\}$. So:

$$\vec{u}_1 = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots \qquad \Rightarrow \quad i_1 \geq j_1.$$

Reversing $U$ & $V$ gives $j_1 \geq i_1$. So $i_1 = j_1$ (and $a_1 = 1$).

(B) Next, $\vec{u}_2 = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots \Rightarrow \vec{u}_2$ has $b_1$ in $(j_1 =) i_1^{\text{th}}$ coord.

$$\Rightarrow b_1 = 0$$
$$\Rightarrow \vec{u}_2 = b_2\vec{v}_2 + b_3\vec{v}_3 + \cdots$$

(and vice versa).

(C) Striking out the first rows of $U$ and $V$, the remaining (still RREF!) matrices are row equivalent. Go back to (A), find $i_2 = j_2$, and repeat until there's nothing left.

## (3) The non-pivot columns of $U$ & $V$ are equal.

(D) We have $\vec{u}_1 = \vec{v}_1 + a_2\vec{v}_2 + \cdots + a_m\vec{v}_m$ (recall $a_1 = 1$ above)

$\Rightarrow \vec{u}_1$ has $a_2$ in the $(j_2 =) i_2^{\text{th}}$ coordinate

$a_3$ in the $(j_3 =) i_3^{\text{th}}$ coordinate, etc

$$\Rightarrow \quad 0 = a_2 = a_3 = \cdots = a_m$$

So $\vec{u}_1 = \vec{v}_1$.

(E) Strike out the first row of $U$ & $V$, go back to (D), get $\vec{u}_2 = \vec{v}_2$, and so on. $\square$