Lecture 30: More on partial derivatives

I just want to convey a couple of ideas so as to leave time for practice exam questions.

(1) What the wave equation has to do with Bessel functions

In 2 dimensions, the wave equation is (for $f = f(x,y,t)$)

$$(*) \qquad \frac{\partial^2 f}{\partial t^2} = c^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

think of a vibrating drum head

Suppose f depends only on t & $r = \sqrt{x^2+y^2}$, and factors as a product $f(x,y,t) = F(\sqrt{x^2+y^2})\,G(t)$. Then

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} F'(\sqrt{x^2+y^2})\,G(t) = \frac{x}{r}F'(r)\,G(t), \quad \frac{\partial f}{\partial y} = \frac{y}{r}F'(r)\,G(t)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} F'(\sqrt{x^2+y^2})\,G(t) + \left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 F''(\sqrt{x^2+y^2})\,G(t)$$

$$= \frac{y^2}{r^3}F'(r)\,G(t) + \frac{x^2}{r^2}F''(r)\,G(t), \quad \frac{\partial^2 f}{\partial y^2} = \frac{x^2}{r^3}F'G + \frac{y^2}{r^2}F''G$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\frac{1}{r}F' + F''\right)G$$

$$\underset{(*)}{\Rightarrow} FG'' = c^2\left(\frac{1}{r}F' + F''\right)G \Rightarrow \overbrace{\frac{G''}{G}}^{\text{cst. in } x} = c^2\overbrace{\left(\frac{\frac{1}{r}F' + F''}{F}\right)}^{\text{cst. in } t}$$

hence $\underline{\underline{\text{constant}}} =: \kappa$

$$\rightarrow G'' = \kappa G, \quad \frac{1}{r}F' + F'' = \frac{\kappa}{c^2}F.$$

e.g. if $\kappa = -c^2$, then $G = a_1\cos(ct) + a_2\sin(ct)$ and

$$r^2F''(r) + rF'(r) + r^2F = 0 \qquad \text{(Bessel's eqn. with } \alpha = 0\text{ !)}$$

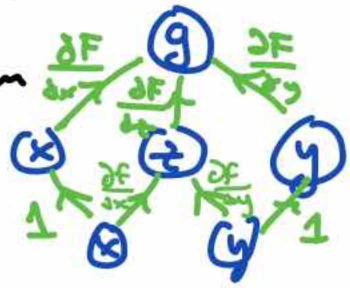
$$\Rightarrow F(r) = c_1 J_0(r) + c_2 K_0(r).$$

② <u>Implicit partial differentiation</u>

Suppose we want to analyze the rates of change of z with respect to x & y in a situation where z depends on these variables through an equation such as

$$\underbrace{y^2 + xz + z^2 - e^z}_{F(x,y,z)} = 4, \qquad \text{(level surface of } F\text{)}$$

that you can't explicitly solve for z. Write formally $z = f(x,y)$ and $g(x,y) := F(x,y,f(x,y))$ (which we will insist remain constant $= 4$). Using the diagram



we obtain

$$\begin{cases} 0 = \dfrac{\partial g}{\partial x} = \dfrac{\partial F}{\partial x}\cdot 1 + \dfrac{\partial F}{\partial z}\cdot\dfrac{\partial f}{\partial x} \\[2mm] 0 = \dfrac{\partial g}{\partial y} = \dfrac{\partial F}{\partial y}\cdot 1 + \dfrac{\partial F}{\partial z}\cdot\dfrac{\partial f}{\partial y} \end{cases}$$

$$\Longrightarrow \frac{\partial f}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z} = \frac{-z}{x+2z-e^z}, \quad \frac{\partial f}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z} = \frac{-2y^2}{x+2z-e^z}.$$

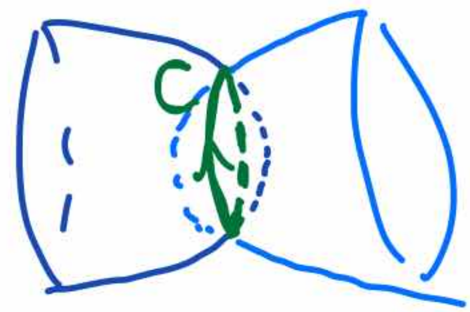
(More informally, $0 = \dfrac{\partial}{\partial y}\left(y^2+xz+z^2-e^z\right) = 2y + x\dfrac{\partial z}{\partial y} + 2z\dfrac{\partial z}{\partial y} - e^z\dfrac{\partial z}{\partial y}$

$\rightarrow \dfrac{\partial z}{\partial y} = \dfrac{2y}{x+2z-e^z}$.)

So at $(0,e,2)$ we get $\dfrac{\partial f}{\partial x} = \dfrac{2}{e^2-4}$, $\dfrac{\partial f}{\partial y} = \dfrac{2e^2}{e^2-4}$

hence tangent vectors $\left(1, 0, \dfrac{2}{e^2-4}\right)$ and $\left(0, 1, \dfrac{2e^2}{e^2-4}\right)$ to $F(x,y,z) = 4$, with cross-product $\left(\dfrac{2}{4-e^2}, \dfrac{2e^2}{4-e^2}, 1\right)$ normal to the surface. This <u>should</u> be parallel to the gradient $\vec{\nabla} F = \left(z, 2y, x+2z-e^z\right)\Big|_{(0,e,2)} = (2, 2e, 4-e^2)$, and it <u>is</u>.

One more computational idea: given two surfaces $F(x,y,z)=0$ and $G(x,y,z)=0$ intersecting in a curve C, let's suppose that on this curve we can parametrize x & y as functions of z: $z \mapsto (X(z), Y(z), z)$. Writing



$f(z) := F(X(z), Y(z), z)$ and $g(z) := G(X(z), Y(z), z)$, staying on the curve C means that f & g remain 0:

$$\begin{cases} 0 = f'(z) = F_x X' + F_y Y' + F_z \\ 0 = g'(z) = G_x X' + G_y Y' + G_z \end{cases} \Rightarrow \begin{cases} F_x X' + F_y Y' = -F_z \\ G_x X' + G_y Y' = -G_z \end{cases}$$

Solve the system for $X'(z)$ & $Y'(z)$ using Cramer:

$$X'(z) = \frac{-\begin{vmatrix} F_z & F_y \\ G_z & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = \frac{-\partial(F,G)/\partial(z,y)}{\partial(F,G)/\partial(x,y)}$$

$$Y'(z) = \frac{-\begin{vmatrix} F_x & F_z \\ G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = \frac{-\partial(F,G)/\partial(x,z)}{\partial(F,G)/\partial(x,y)}.$$

Apostol has a section 9.7 consisting of "worked examples" of such things. This is a reading assignment!