Lecture 30: More on partial derivatives

I just want to convey a couple of ideas so as to leave time for practice exam questions.

1. What the wave equation has to do with Bessel functions

In 2 dimensions, the wave equation is (for \( f = f(x, y, t) \))

\[
\frac{\partial^2 f}{\partial t^2} = c^2 \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).
\]

Then, if \( f \) depends only on \( t \) and \( r = \sqrt{x^2 + y^2} \), and factors as a product \( f(x, y, t) = F(\sqrt{x^2 + y^2})G(t) \). Then

\[
\frac{df}{dx} = \frac{x}{\sqrt{x^2 + y^2}} F'(\sqrt{x^2 + y^2})G(t) = \frac{x}{r} F'(r)G(t), \quad \frac{df}{dy} = \frac{y}{r} F'(r)G(t)
\]

\[
\frac{d^2 f}{dx^2} = \frac{x^2}{x^2 + y^2} F'(\sqrt{x^2 + y^2})G(t) + \left( \frac{r}{\sqrt{x^2 + y^2}} \right)^2 F''(\sqrt{x^2 + y^2})G(t)
\]

\[
= \frac{x^2}{r^2} F'(r)G(t) + \frac{x^2}{r} F''(r)G(t), \quad \frac{d^2 f}{dy^2} = \frac{x^2}{r^2} F'(r)G(t) + \frac{x^2}{r} F''(r)G(t)
\]

\[
\Rightarrow \frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \left( \frac{1}{r} F' + F'' \right)G
\]

\[
FG'' = c^2 \left( \frac{1}{r} F' + F'' \right)G \Rightarrow \frac{G''}{G} = c^2 \left( \frac{1}{r} F' + F'' \right)
\]

\[
\Rightarrow G'' = \kappa G, \quad \frac{1}{r} F' + F'' = \frac{\kappa}{c^2} F.
\]

E.g. if \( \kappa = -c^2 \), then \( G = a_1 \cos(\omega t) + a_2 \sin(\omega t) \) and

\[
r^2 F''(r) + r F'(r) + F(r) = 0 \quad (\text{Bessel's eqn. with } a = 0 !)
\]

\[
\Rightarrow F(r) = c_1 J_0(r) + c_2 K_0(r).
\]
② Implicit partial differentiation

Suppose we want to analyze the rates of change of $z$ with respect to $x$ and $y$ in a situation where $z$ depends on these variables through an equation such as

$$y^2 + x^2 + z^2 - e^z = 4, \quad (\text{level surface of } F)$$

that you can't explicitly solve for $z$. Write formally $z = f(x,y)$ and $g(x,y) := F(x,y, f(x,y))$ (which we will insist remain constant $z = 4$). Using the diagram we obtain

$$0 = \frac{\partial g}{\partial x} = \frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial x}, \quad 0 = \frac{\partial g}{\partial y} = \frac{\partial F}{\partial y} \cdot 1 + \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \frac{-\frac{\partial F}{\partial x} \cdot 1 - \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial x}}{\frac{\partial F}{\partial y} \cdot 1 + \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial y}} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = \frac{-2y^2}{x + 2z - e^z}. \quad \frac{\partial f}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{-2y^2}{x + 2z - e^z}.$$

(More informally, $0 = \frac{\partial}{\partial y}(y^2 + x^2 + z^2 - e^z) = 2y + x \frac{\partial}{\partial y}z + 2z \frac{\partial}{\partial y}e^z - e^z \frac{\partial}{\partial y}$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{2y}{x + 2z - e^z}$$

So at $(0, e, 2)$ we get $\frac{\partial f}{\partial x} = \frac{2}{e^2 - 4}, \quad \frac{\partial f}{\partial y} = \frac{2e^2}{e^2 - 4}$

hence tangent vectors $(1, 0, \frac{2}{e^2 - 4})$ and $(0, 1, \frac{2e^2}{e^2 - 4})$

to $F(x,y,z) = 4$, with cross-product $(\frac{2}{4e^2}, \frac{2e^2}{e^2 - 4}, 1)$

normal to the surface. Thus should be parallel to the gradient $\nabla F = (2, 2y, x + 2z - e^z)|_{(0,e,2)} = (2, 2e, 4e^2 - 4)$, and it is.
One more computational idea: given two surfaces \( F(x,y,z) = 0 \) and \( G(x,y,z) = 0 \) intersecting in a curve \( C \), let's suppose that on this curve we can parameterize \( x \) and \( y \) as functions of \( t \):

\[ t \mapsto (X(t), Y(t), z). \]

Writing \( f(t) := F(X(t), Y(t), z) \) and \( g(t) := G(X(t), Y(t), z) \), staying on the curve \( C \) means that \( f \) and \( g \) remain 0:

\[
\begin{align*}
0 &= f'(t) = F_x X' + F_y Y' + F_z = \frac{\partial F}{\partial x} X' + \frac{\partial F}{\partial y} Y' + F_z = \frac{\partial F}{\partial x} X' + \frac{\partial F}{\partial y} Y' - F_z, \\
0 &= g'(t) = G_x X' + G_y Y' + G_z = \frac{\partial G}{\partial x} X' + \frac{\partial G}{\partial y} Y' + G_z = \frac{\partial G}{\partial x} X' + \frac{\partial G}{\partial y} Y' - G_z.
\end{align*}
\]

Solve the system for \( X'(t) \) & \( Y'(t) \) using Cramer:

\[
X'(t) = - \frac{| F_x F_y |}{| G_x G_y |} = - \frac{\partial F / \partial y}{\partial G / \partial y}.
\]

\[
Y'(t) = - \frac{| F_x F_y |}{| G_x G_y |} = - \frac{\partial F / \partial x}{\partial G / \partial x}.
\]

Apostol has a section 9.7 consisting of "worked examples" of such things. This is a reading assignment!