Lecture 31: Extrema of multivariable functions

Setup $\vec{a} \in S \xrightarrow{f} \mathbb{R}$ function of n variables $(x_1, \dots, x_n)$

point $\quad S \subset \mathbb{R}^n$ subset

Definition: • f has a (global/absolute) $\overset{\text{minimum}}{\text{maximum}}$ at $\vec{a}$

if $f(\vec{a}) \overset{\le}{\ge} f(\vec{x})$ for all $\vec{x} \in S$.

(Then $f(\vec{a})$ is called the $\overset{\text{minimum}}{\text{maximum}}$ value.)

• f has a relative/local $\overset{\text{minimum}}{\text{maximum}}$ at $\vec{a}$

if $f|_{B(\vec{a};\epsilon) \cap S}$ has a $\overset{\text{minimum}}{\text{maximum}}$ at $\vec{a}$

for $\epsilon > 0$ sufficiently small.

the restriction of f to $B(\vec{a};\epsilon) \cap S$

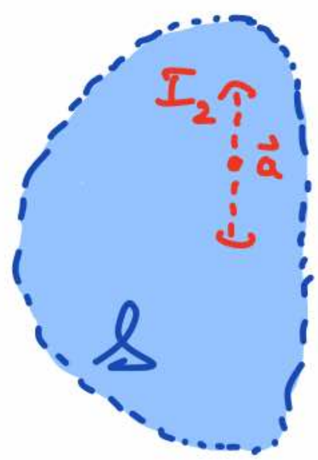
• an extremum (global or local) of f means

a maximum or minimum of f (global or local)

Theorem 1: Assume $\vec{a} \in \operatorname{int}(S)$. If f has a local extremum at $\vec{a}$, then $\vec{\nabla} f(\vec{a}) = \vec{0}$, i.e. $\vec{a}$ is a stationary point of f.

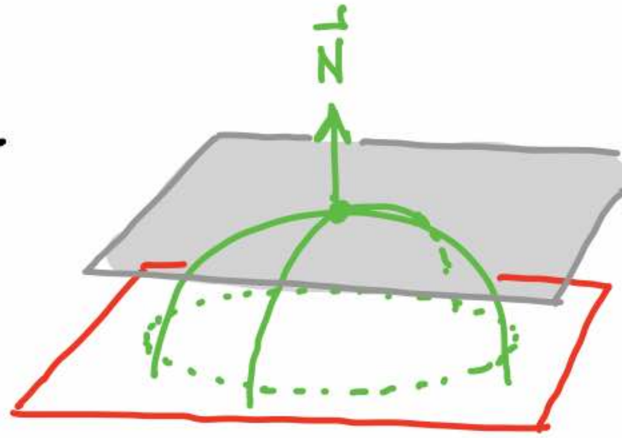
Proof: Write $\vec{a} = (a_1, \dots, a_n)$, and restrict f to the interval $I_j = \{(a_1, \dots, x_j, \dots, a_n) \mid a_j - \epsilon < x_j < a_j + \epsilon\}$:

This is a function of a single variable x_j, call it $F(x_j) := f(a_1, \dots, x_j, \dots, a_n)$, with max/min at $(x_j =)\ a_j$.



A theorem from Math 203 says that $F'(a_j) = 0$.

But $F'(a_j)$ is $\frac{\partial f}{\partial x_j}(\vec{a})$, which is therefore 0 for each j. $\square$



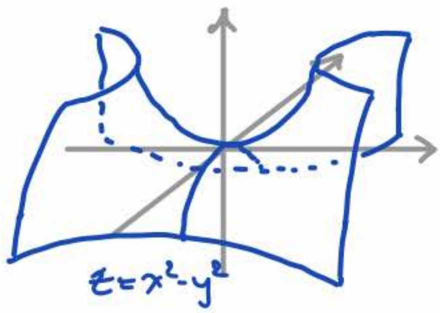
Remarks: (A) At a stationary point, the tangent plane to the graph of

$(*)$ $\quad z = f(x_1, \dots, x_n)$

in $\mathbb{R}^{n+1}$ is $z = f(\vec{a})$ ("horizontal").

Why? The normal to $(*)$ is $\vec{\nabla}(z - f(\vec{x})) = (-\vec{\nabla} f(\vec{a}), 1) = (\vec{0}, 1)$.

(B) The converse to Theorem 1 is FALSE:



$f(x,y) = x^2 - y^2$ has $\vec{\nabla} f(0,0) = \vec{0}$, but is

(instead of x_1, x_2)

concave $\begin{cases} \text{up} \\ \text{down} \end{cases}$ in the $\begin{cases} x \\ y \end{cases}$ -direction.

A saddle-point is a stationary point which is not a local extremum.

$\boxed{\text{EXAMPLE}}$ $\quad z = f(x,y) = x^2 + 4xy + y^2$ (what can we say about it?)

- xz-cross-section: $f(x,0) = x^2 \rightsquigarrow$ concave up
- yz-cross-section: $f(0,y) = y^2 \rightsquigarrow$ also concave up

$\left.\right\} f_{xx}(0) = 2 = f_{yy}(0)$

- Also $D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = u_x f_x + u_y f_y = u_x(2x + 4y) + u_y(2y + 4x)$

  is 0 at $\vec{0}$, so $(0,0)$ is certainly a stationary point of f

$\rightsquigarrow$ suggests local minimum at $(0,0)$? Just to be sure, let's

- differentiate twice in the "SE-direction":

$$D^2_{\hat{u}} f = D_{\hat{u}}(D_{\hat{u}} f) = D_{\hat{u}}\left\{\tfrac{\sqrt{2}}{2}\left(2x + 4y - (2y + 4x)\right)\right\} = D_{\hat{u}}\left\{\sqrt{2}(y - x)\right\}$$

$$= \vec{\nabla}(\sqrt{2}(y-x)) \cdot \tfrac{\sqrt{2}}{2}(1,-1) = (-\sqrt{2}, \sqrt{2}) \cdot \tfrac{1}{\sqrt{2}}(1,-1) = \underline{-2} \quad (!!)$$

<u>Moral</u>: finding the concavities in the x- & y-directions may not tell you much about the directions of greatest or least concavity, or what those greatest & least concavities are!

To explore further, let $\vec{a}$ be a stationary point of a function $f(x,y)$, and set $\hat{u} = (\overset{u_x}{\cos\theta}, \overset{u_y}{\sin\theta})$. Then

$$\begin{aligned}
D^2_{\hat{u}} f &= D_{\hat{u}}\left(u_x f_x + u_y f_y\right) = \vec{\nabla}(u_x f_x + u_y f_y)\cdot \hat{u} \\
&= (u_x f_{xx} + u_y f_{yx},\ u_x f_{xy} + u_y f_{yy})\cdot(u_x, u_y) \\
&= (u_x, u_y)\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} =: \hat{u}^T H \hat{u} = \hat{u}\cdot H\hat{u},
\end{aligned}$$

where the 2×2-matrix-valued function $H(x,y)$ is called the <u>Hessian</u> of f. Write H_0 for $H(\vec{a})$.

So $\hat{u}\cdot H\hat{u}$ is the concavity at $\vec{a}$ in the direction $\hat{u}$, which depends on θ. We want the directions of extreme concavity:

$$0 = \frac{d}{d\theta}\hat{u}\cdot H_0\hat{u} = \hat{u}'\cdot H_0\hat{u} + \hat{u}\cdot H_0\hat{u}' = 2\hat{u}'\cdot H_0\hat{u}$$

$$\Longrightarrow \hat{u}' \perp H_0\hat{u} \quad \overset{\dagger}{\Longrightarrow} \quad \hat{u} \parallel H_0\hat{u} \quad \Longrightarrow \quad \underline{\hat{u} \text{ is an eigenvector of } H_0\,!}$$

If $\hat{u} = \vec{v} :=$ eigenvector w/eigenvalue λ, then

$$D^2_{\hat{u}} f = \vec{v}\cdot H_0\vec{v} = \vec{v}\cdot\lambda\vec{v} = \lambda\cancelto{1}{\|\vec{v}\|^2} = \underline{\lambda \text{ is the concavity}}\,!$$

† $\hat{u}' = \frac{d}{d\theta}(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta)$ is $\perp$ to $\hat{u} = (\sin\theta, \cos\theta)$

Writing $\lambda_1 \le \lambda_2$ for the 2 eigenvalues, these are the minimum & maximum concavities of f at $\vec{a}$ $\Longrightarrow$

FACT 1 $\Delta := \det \begin{pmatrix} f_{xx}(\vec{a}) & f_{xy}(\vec{a}) \\ f_{yx}(\vec{a}) & f_{yy}(\vec{a}) \end{pmatrix}$ is the product of max + min concavities at $\vec{a}$.

But if $\Delta > 0$, this tells me only that $\lambda_1, \lambda_2 > 0$ OR $\lambda_1, \lambda_2 < 0$. Which is it?

Consider $f_{xx}(\vec{a})$, the concavity in the x-direction — as such, clearly in between the max/min concavities:

$$\lambda_1 \le f_{xx}(\vec{a}) \le \lambda_2 .$$

So if $f_{xx}(\vec{a}) > 0$, then $\lambda_2 > 0$; and since we assumed $\Delta = \lambda_1 \lambda_2 > 0$, $\lambda_1 > 0$ too! $\Longrightarrow$

FACT 2

(2nd derivative test)

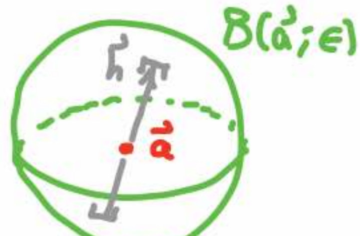
	Δ	$f_{xx}(\vec{a})$	Consequence
1	>0	>0	local min
2	>0	<0	local max
3	<0	—	saddle = stationary point which is not a local extremum
	0	—	inconclusive

To recap: the 2 facts are for use in the case where f is twice continuously differentiable in a neighborhood of $\vec{a}$, with $\vec{\nabla} f(\vec{a}) = \vec{0}$. 1, 2, 3 correspond to the quadratic form $Q(\vec{h}) = \vec{h}^T H(\vec{a}) \vec{h}$ being 1 positive definite / 2 negative definite / 3 indefinite.

$\boxed{\text{EXAMPLE (cont'd.)}}$ $\quad \vec{a} = \vec{0}, \quad f(x,y) = x^2 + 4xy + y^2$

$$H(x,y) = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}, \quad \Delta = 4 - 16 = -12 < 0$$

$\Rightarrow$ saddle point.



Now we generalize to n variables and examine $f(\vec{x})$ on the gray axis in the picture:

$g(u) := f(\vec{a} + u\vec{h})$ for $u \in [-1,1]$.

$\Downarrow$ Chain rule

$$g'(u) = \vec{\nabla} f(\vec{a} + u\vec{h}) \cdot \vec{h} = \sum_{j=1}^{n} h_j f_{x_j}(\vec{a} + u\vec{h})$$

$\Downarrow$ Chain rule

$$g''(u) = \vec{\nabla}\Big(\sum_j h_j f_{x_j}(\vec{a} + u\vec{h})\Big) \cdot \vec{h} = \sum_{i,j} h_i h_j f_{x_j x_i}(\vec{a} + u\vec{h})$$

$$= \vec{h}^T H(\vec{a} + u\vec{h})\,\vec{h}, \quad \text{where } H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right] =: \text{Hessian of } f$$

$\Downarrow$

Taylor remainder: $c \in (0,1)$

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = g(1) - g(0) = g'(0) + \tfrac{1}{2} g''(c)$$

$$= \vec{\nabla} f(\vec{a}) \cdot \vec{h} + \tfrac{1}{2} \vec{h}^T H(\vec{a} + c\vec{h})\,\vec{h}$$

(**) $$= \vec{\nabla} f(\vec{a}) \cdot \vec{h} + \tfrac{1}{2} \vec{h}^T H(\vec{a})\,\vec{h} + \|\vec{h}\|^2 E_2(\vec{a};\vec{h}) \quad (\dagger)$$

($\dagger$) where $E_2(\vec{a};\vec{h}) = \begin{cases} \vec{h}^T \dfrac{H(\vec{a}+c\vec{h}) - H(\vec{a})}{2\|\vec{h}\|^2}\,\vec{h}, & \vec{h} \neq 0 \\ 0, & \vec{h} = \vec{0} \end{cases}$

has $$|E_2(\vec{a};\vec{h})| = \frac{1}{2\|\vec{h}\|^2} \left| \sum_i \sum_j h_i h_j \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}+c\vec{h}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \right) \right|$$

$$\leq \frac{1}{2\|\vec{h}\|^2} \sum_i \sum_j |h_i||h_j| \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}+c\vec{h}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \right|$$

$$\leq \frac{\cancel{\|\vec{h}\|^2}}{2\cancel{\|\vec{h}\|^2}} \sum_i \sum_j \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}+c\vec{h}) - \frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{a}) \right| \longrightarrow 0 \text{ as } \vec{h} \to \vec{0}$$

by continuity of 2nd partials.

where $E_2(\vec{a};\vec{h})$ is $o(1)$ $\left(\to 0 \text{ as } \vec{h}\to\vec{0}\right)$ assuming f is twice continuously differentiable on $B(\vec{a};\epsilon)$.

Now recall from Lecture 15 the

<u>Lemma on Quadratic Forms</u>: Let $Q(\vec{h}) = \vec{h}^T A \vec{h}$, A symmetric $n\times n$ with eigenvalues $\lambda_1 \le \cdots \le \lambda_n$. Then

(i) $Q(\vec{h}) > 0$ $\forall \vec{h}\neq\vec{0}$ $\iff$ all $\lambda_j > 0$ $\iff$ A positive-definite

(ii) $Q(\vec{h}) < 0$ $\forall \vec{h}\neq\vec{0}$ $\iff$ all $\lambda_j < 0$ $\iff$ A negative-definite

(iii) $Q(\vec{h})$ takes $+/-$ values $\iff$ $\lambda_1 < 0 < \lambda_n$ $\iff$ A indefinite.

<u>Theorem 2</u>: If f is twice continuously differentiable, with a stationary point at $\vec{x}=\vec{a}$, then:

(i) $H(\vec{a})$ pos. definite $\implies$ f has rel. minimum at $\vec{a}$.

(ii) $H(\vec{a})$ neg. definite $\implies$ f has rel. maximum at $\vec{a}$.

(iii) $H(\vec{a})$ indefinite $\implies$ f has saddle point at $\vec{a}$.

<u>Proof</u>: Since $\vec{\nabla} f(\vec{a}) = \vec{0}$ at a stationary point, $(**)$ becomes

$$f(\vec{a}+\vec{h}) - f(\vec{a}) = \vec{h}^T H(\vec{a})\,\vec{h} + \underbrace{\|\vec{h}\|^2 E_2(\vec{a};\vec{h})}.$$

The main idea is that this term doesn't matter, and so we are done by the lemma.

Here is how this works for (i):

- let $0 \le \lambda_1 \le \cdots \le \lambda_n$ be eigenvalues of $H(\vec{a})$.

  Pick $\epsilon > 0$ s.t. $|E_2(\vec{a};\vec{h})| < \frac{1}{4}\lambda_1$ for $0 < \|\vec{h}\| < \epsilon$.

- let $u \in (0, \lambda_1)$, so that the eigenvalues $\lambda_j - u$ of $H(\vec{a}) - u I_n$ are all positive. Lemma $\Rightarrow$ $\underbrace{\vec{h}^T [H(\vec{a}) - u I_n] \vec{h}}_{Q(\vec{h})} > 0 \; \forall \vec{h} \ne \vec{0}$.

- so $\vec{h}^T H(\vec{a}) \vec{h} > u \|\vec{h}\|^2 \; \forall \vec{h} \ne \vec{0}$, and taking $u = \frac{1}{2}\lambda_1$ gives

  $\frac{1}{2}\vec{h}^T H(\vec{a}) \vec{h} > \frac{1}{4}\lambda_1 \|\vec{h}\|^2 > \|\vec{h}\|^2 |E_2(\vec{a};\vec{h})| \geqslant 0$

  $$\Rightarrow f(\vec{a}+\vec{h}) - f(\vec{a}) = \frac{1}{2}\vec{h}^T H(\vec{a}) \vec{h} + \|\vec{h}\|^2 |E_2(\vec{a};\vec{h})|$$

  $$\geqslant \frac{1}{2}\vec{h}^T H(\vec{a}) \vec{h} - \|\vec{h}\|^2 |E_2(\vec{a};\vec{h})| > 0 \quad \text{for } \vec{h} \ne \vec{0},$$

  and f has a relative minimum at $\vec{a}$. $\square$