Lecture 32

Optimization
Yesterday we were mainly concerned with local extrema at interior points of a set, showing that for a differentiable function there must be stationary points (where the gradient = 0). What about global extrema?
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What about global extrema?

**Theorem 1:** (i) If $S \subseteq \mathbb{R}^n$ is a closed, bounded set, and $f$ is continuous on $S$, then it attains (absolute/global) max + min on $S$.

(ii) These extrema can occur at
- $\partial S = \text{boundary of } S$ (by which I mean it is not differentiable on a neighborhood of the point)
- singular points of $f(x)$
- stationary points of $f(x)$ in $\text{int}(S)$ (by defn.)
Yesterday we were mainly concerned with local extrema at interior points of a set, showing that — for a differentiable function — there must be stationary points (where the gradient = 0). What about global extrema?

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- stationary points of \( f(x) \).

I defer the proof of (i) (in a special case) for now, and only observe that (ii) is an immediate consequence of yesterday's result.
Yesterday we were mainly concerned with local extrema at interior points of a set, showing that — for a differentiable function — there must be stationary points (where the gradient = 0). What about global extrema?

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- \( \partial S = \) boundary of \( S \)
- **singular points of** \( f(x) \)
- stationary points \( f(x) \)

I defer the proof of (i) (in a special case) for now, and only observe that (i) is an immediate consequence of yesterday's results: let \( S_0 \subseteq S \) be the interior of the subset on which \( f \) is differentiable, and let \( \hat{a} \) be an extremum of \( f \). If \( \hat{a} \in S_0 \), we know \( \nabla f(\hat{a}) \) is zero. Otherwise, either \( \hat{a} \in \partial S \) or \( \hat{a} \in \text{int}(S) \backslash S_0 = \) singular points.
Yesterday we were mainly concerned with local extrema at interior points of a set, showing that for a differentiable function there must be stationary points (where the gradient = 0). What about global extrema?

**Theorem 1:** (i) If \( S \subseteq \mathbb{R}^n \) is a closed, bounded set, and \( f \) is continuous on \( S \), then it attains (absolute/global) max + min on \( S \).

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- \( \partial S \) = boundary of \( S \) (by which I mean \( f \) is not differentiable on a neighborhood of the point)
- Singular points of \( f(x) \)
- Stationary points of \( f(x) \).

I defer the proof of (i) (in a special case) for now, and only observe that (ii) is an immediate consequence of yesterday's results:

Let \( S_0 \subseteq S \) be the interior of the subset on which \( f \) is differentiable, and let \( \hat{a} \) be an extremum of \( f \). If \( \hat{a} \in S_0 \), we know \( \nabla f(\hat{a}) \) is zero. Otherwise, either \( \hat{a} \in \partial S \) or \( \hat{a} \in \text{int}(S) \setminus S_0 \) = singular points.

**Upshot:** We get an algorithm for locating \( f \)'s extrema, known as optimizing \( f \):

1. **Step 1:** Minimize/maximize \( f \) on \( \partial S \)
2. **Step 2:** Evaluate \( f \) at stationary points (excluding obvious bad points)
3. **Step 3:** Pick the biggest/smallest from Steps 1 & 2.
Ex 1) Optimize \( f(x, y) = 8xy - x - y \) on the closed triangular region with vertices \((0,2)\), \((2,0)\), and \((0,0)\).
Ex 1) Optimize \( f(x, y) = 8xy - x - y \) on the closed triangular region.

- \( \overline{AC} : y = 0 \). \( f(x, 0) = -x \) \( \min \) at \( C : -2 \)
- \( \overline{AB} : x = 0 \). \( f(0, y) = -y \) \( \max \) at \( A : 0 \)
- \( \overline{BC} : x = 2 \). \( f(2, y) = 16y - 2 - y \)
Ex 1] Optimize \( f(x, y) = 8xy - x - y \) on the closed triangular region.

- \( \overline{AC} : y = 0. \) \( f(x, 0) = -x \) max at \( A : 0 \)
- \( \overline{AB} : x = 0. \) \( f(0, y) = -y \) min at \( B : -2 \)
- \( \overline{BC} : f(2-t, t) = 16t - 8t^2 - 2 \)

\[ D = \frac{\partial}{\partial x} f(2-t, t) = 16 - 16t \text{ given } t = 1, \]
and \( f(1, 1) = 6. \)
Ex 1) Optimize $f(x,y) = 8xy - x - y$ on the closed triangular region.

- $\overline{AC} : y = 0$. $f(x,0) = -x \Rightarrow \max \text{ at } A : 0$
- $\overline{AB} : x = 0$. $f(0,y) = -y \Rightarrow \min \text{ at } B : -2$
- $\overline{BC} : f(2-t,t) = 16t - 8t^2 - 2$
  \[ \Rightarrow 0 = \frac{\partial}{\partial x} f(2-t,t) = 16 - 16t \text{ give } t=1, \]
  and $f(1,1) = 6$.
- interior: $\vec{0} = \vec{\nabla} f = (8y-1, 8x-1)$
  \[ \Rightarrow (x,y) = \left( \frac{1}{8}, \frac{1}{8} \right), \text{ at which } f = -\frac{1}{8}. \]
  (it's a saddle point, but we don't care!)

Conclude that overall
\[ \max = 6, \text{ at } (1,1) \]
\[ \min = -2, \text{ at } B \& C \]
Ex 1] Optimize \( f(x,y) = 8xy - x - y \) on the closed triangular region 

![Diagram of a triangle with vertices A(0,0), B(2,0), and C(0,2)]

- \( \overline{AC} : y = 0 \) \( \Rightarrow f(x,0) = -x \), max at A: 0
- \( \overline{AB} : x = 0 \) \( \Rightarrow f(0,y) = -y \), min at C: -2
- \( \overline{BC} : f(2-x,t) = 16t - 8t^2 - 2 \)
  \( \Rightarrow \) 0 = \( \frac{d}{dx} f(2-x,t) = 16-16t \) giving \( t = 1 \), and \( f(1,1) = 6 \).
- interior: \( \vec{0} = \nabla f = (8y-1, 8x-1) \)
  \( \Rightarrow (x,y) = \left( \frac{1}{8}, \frac{1}{8} \right) \), at which \( f = -\frac{1}{8} \).
  (it's a saddle point, but we don't care!)

Conclude that overall:
- \( \max = 6 \), at (1,1)
- \( \min = -2 \), at B & C

Ex 2] Optimize \( f(x,y) = 2x^2 + y^2 - 4x - 2y + 5 \) on the closed set \( S = \{(x,y) | x^2 + \frac{y^2}{2} \leq 1 \} \).
Ex 1] Optimize $f(x, y) = 8x y - x - y$ on the closed triangular region $\triangle ABC$.

- $\overline{AC} : y = 0$. $f(x, 0) = -x \quad \text{max at } A : 0$
- $\overline{AB} : x = 0$. $f(0, y) = -y \quad \text{min at } A : 0$
- $\overline{BC} : f(2-t, t) = 16t - 8t^2 - 2$
  $\Rightarrow 0 = \frac{\partial}{\partial x} f(2-t, t) = 16 - 16t \quad \text{gives } t = 1,$
  and $f(1, 1) = 6$
- Interior: $\vec{0} = \nabla f = (8y - 1, 8x - 1)$
  $\Rightarrow (x, y) = \left(\frac{1}{8}, \frac{1}{8}\right)$, at which $f = -\frac{1}{8}$.
  (it's a saddle point, but we don't care!)

Conclude that overall:
- $\text{max} = 6$, at $(1, 1)$
- $\text{min} = -2$, at $B$ & $C$

Ex 2] Optimize $f(x, y) = 2x^2 + y^2 - 4x - 2y + 5$ on the closed set $S = \{(x, y) \mid x^2 + \frac{y^2}{2} \leq 1\}$.

- $\partial S = \text{ellipse, parametrized by } (x, y) = (\cos t, \sqrt{2}\sin t)$

Write $F(t) = f(x(t), y(t))$, so that Chain Rule $\Rightarrow$

$F'(t) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)$

$= (4x(t) - 4)(-\sin t) + (2y(t) - 2)(\sqrt{2}\cos t)$

$= (4\cos t - 4)(-\sin t) + (2\sqrt{2}\sin t - 2)(\sqrt{2}\cos t)$

$= 4\sqrt{2}\sin t - 2\sqrt{2} \cos t$.  

**Ex 1**) Optimize \( f(x,y) = 8xy - x - y \) on the closed triangular region \( \triangle ABC \).

- \( \overline{AC} : y = 0 \). \( f(x,0) = -x \) \( \max \) at \( A : 0 \).
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- \( \overline{BC} : f(2-t,t) = 16t - 8t^2 - 2 \)
  \[ \Rightarrow 0 = \frac{d}{dt} f(2-t,t) = 16 - 16t \text{ given } t=1, \]
  and \( f(1,1) = 6 \).
- Interior: \( \nabla f = (8y-1, 8x-1) \)
  \[ \Rightarrow (x,y) = \left( \frac{1}{8}, \frac{1}{8} \right), \text{ at which } f = -\frac{1}{8} \).
  (it's a saddle point, but we don't care.)

Conclude that overall \( \max = 6 \), at \( (1,1) \)
\( \min = -2 \), at \( B \& C \).

**Ex 2**) Optimize \( f(x,y) = 2x^2 + y^2 - 4x - 2y + 5 \) on the closed set \( \mathcal{S} = \{(x,y) \mid x^2 + \frac{y^2}{4} \leq 1\} \).

- \( \partial \mathcal{S} = \) ellipse, parameterized by \( (x,y) = (\cos t, \frac{\sqrt{4}}{2} \sin t) \)

  Write \( F(t) = f(x(t), y(t)) \), so that chain rule \( \Rightarrow \)
  \[ F'(t) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t) \]
  \[ = (4x(t) - 4)(-\sin t) + (2y(t) - 2)(\frac{\sqrt{4}}{2} \cos t) \]
  \[ = (4 \cos t - 4)(-\sin t) + (2 \frac{\sqrt{4}}{2} \sin t - 2)(\frac{\sqrt{4}}{2} \cos t) \]
  \[ = 4 \sin t - 2 \sqrt{4} \cos t . \]

  Setting \( 0 = F'(t) \) gives \( \tan t = \frac{\sqrt{2}}{2} \)
  \[ \Rightarrow x(t) = \cos (\arccos(\frac{\sqrt{2}}{2})) = \frac{-2}{\sqrt{2}} \]
  \[ y(t) = \sqrt{2} \sin (\arccos (\frac{\sqrt{2}}{2})) = \frac{-2}{\sqrt{2}} \]
  \[ f\left( \frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}} \right) = 7 - 2 \sqrt{2}, \quad f\left( \frac{-2}{\sqrt{2}}, \frac{-2}{\sqrt{2}} \right) = 7 + 2 \sqrt{2} \]
**Ex 1**] Optimize $f(x,y) = 8xy - x - y$ on the closed triangular region

- $\overline{AC} : y = 0$. $f(x,0) = -x \Rightarrow \max$ at $A : 0$
- $\overline{AB} : x = 0$. $f(0,y) = -y \Rightarrow \min$ at $B : -2$
- $\overline{BC} : f(2-t, t) = 16t - 8t^2 - 2$
  $\Rightarrow 0 = \frac{\partial f}{\partial x}(2-t, t) = 16 - 16t$ gives $t = 1$, and $f(1,1) = 6$.
- Interior: $\hat{0} = \nabla f = (8y - 1, 8x - 1)$
  $\Rightarrow (x, y) = \left(\frac{1}{8}, \frac{1}{8}\right)$, at which $f = -\frac{1}{8}$.
  (it's a saddle point, but we don't care!)

Conclude that overall

- $\max = 6$, at $(1,1)$
- $\min = -2$, at $B \& C$

**Ex 2**] Optimize $f(x,y) = 2x^2 + y^2 - 4x - 2y + 5$ on the closed set $\mathcal{S} = \{(x,y) \mid x^2 + \frac{y^2}{4} \leq 1\}$.

- $\partial \mathcal{S} =$ ellipse, parametrized by $(x, y) = (\cos \theta, \sqrt{1 - \cos^2 \theta})$
  Write $F(t) = f(x(t), y(t))$, so that chain rule $\Rightarrow$
  $F'(t) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)$
  $= (4x(t) - 4)(-\sin \theta) + (2y(t) - 2)(\cos \theta)$
  $= (4 \cos \theta - 4)(-\sin \theta) + (2 \sqrt{1 - \cos^2 \theta} - 2)(\cos \theta)$
  $= 4 \sin \theta - 2 \sqrt{1 - \cos^2 \theta}$.

Setting $\partial F(t)$ gives $\tan \theta = \frac{\sqrt{2}}{2}$
  $\Rightarrow x(t) = \cos \left( \arctan \left( \frac{\sqrt{2}}{2} \right) \right) = -\frac{\sqrt{2}}{2}$
  $y(t) = \sqrt{1 - \left( \frac{\sqrt{2}}{2} \right)^2} = -\frac{\sqrt{2}}{2}$
  $f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 7 - 2\sqrt{2}$, $f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = 7 + 2\sqrt{2}$

- interior: $\hat{0} = \nabla f = (4x - 4, 2y - 2) \Rightarrow (x, y) = (1, 1)$
  ... and $f(1,1) = 2$ ($< 7 - 2\sqrt{2}$) is the minimum...
Ex 1) Optimize $f(x,y) = 8xy - x - y$ on the closed triangular region $\triangle ABC$.

- $AC: y = 0$. $f(x,0) = -x \quad \min$ at $A: 0$
- $AB: x = 0$. $f(0,y) = -y \quad \max$ at $B: 0$
- $BC: f(2-t,t) = 16t - 8t^2 - 2$
  $\Rightarrow \Delta = \frac{1}{2} f(2-t,t) = 16 - 16t$ gives $t = 1$,
  and $f(1,1) = 6$.
- inner: $\hat{0} = \nabla f = (8y-1, 8x-1)$
  $\Rightarrow (x,y) = \left( \frac{1}{8}, \frac{1}{8} \right)$, at which $f = -\frac{1}{8}$.
  (it's a saddle point, but we don't care!)

Conclude that overall $\min = 0$, at $(1,1)$
$\min = -2$, at $B$ & $C$

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Ex 2) Optimize $f(x,y) = 2x^2 + y^2 - 4x - 2y + 5$ on the closed set $\mathcal{S} = \{(x,y) \mid x^2 + \frac{y^2}{2} \leq 1\}$.

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$F'(t) = f_x(x(t),y(t)) x'(t) + f_y(x(t),y(t)) y'(t)$
$= (4x(t) - 4)(-\sin \theta) + (2y(t) - 2)(\sqrt{2}\cos \theta)$
$= (4\cos \theta - 4)(-\sin \theta) + (2\sqrt{2}\sin \theta - 2)(\sqrt{2}\cos \theta)$
$= 4\sin \theta - 2\sqrt{2} \cos \theta$.

Setting $0 = F'(t)$ gives $\tan \theta = \frac{\sqrt{2}}{2}$
$\Rightarrow x(t) = \cos (\arctan \left( \frac{\sqrt{2}}{2} \right)) = -\frac{2}{\sqrt{2}}$
$y(t) = \sqrt{2} \sin (\arctan \left( \frac{\sqrt{2}}{2} \right)) = \frac{2}{\sqrt{2}}$

$f(\frac{2}{\sqrt{2}}, \frac{2}{\sqrt{2}}) = 7 - 2\sqrt{2}$, $f(\frac{-2}{\sqrt{2}}, \frac{2}{\sqrt{2}}) = 7 + 2\sqrt{2}$

- inner: $\hat{0} = \nabla f = (4x-4, 2y-2) \Rightarrow (x,y) = (1,1)$
  ... and $f(1,1) = 2 \left( < 7 - 2\sqrt{2} \right)$ is the minimum.

NOOO! $(1,1)$ is not in $\mathcal{S}$!
So $7 \pm 2\sqrt{2}$ are the max/min values.
Ex 3: Find 3 positive numbers that add up to 120 and such that their product is a maximum.
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What are the set D & function f?

Want to maximize $xyz$ on $\begin{cases} x+y+z = 120 \\ x,y,z \geq 0 \end{cases}$

i.e. $f(x,y) = xy(120-x-y)$ on $D = \{(x,y) : (0,120), (120,0), (0,0)\}$

$f$ is $\leq$ on $D$. 


Ex. 3 | Find 3 positive numbers that add up to 120 & such that their product is a maximum.

What are the set & function $f$?

Want to maximize $xyz$ on $\{x+y+z=120, \quad x, y, z \geq 0\},$

i.e. $f(x, y) = xy(120-x-y)$ on $\triangle (0,0,0), (0,120), (120,0)$

$f$ is $\leq$ on $\partial \triangle$.

So consider $\delta = \nabla f = \nabla (120xy - x^2 - y^2)$

$\implies (0,0) = (120 - 2x - y, 120 - 2x - y, x, y)$

$\implies y(120 - 2x - y) = 0 = x(120 - 2y - x)$.

$\implies 120 - 2x - y = 0 = 120 - 2y - x$

(don't want $x = 0$ or $y = 0$) $\implies y = x$
Ex 3  Find 3 positive numbers that add up to 120 and such that their product is a maximum.

What are the set A & function f?

Want to maximize $xyz$ on $\begin{cases} x + y + z = 120 \\ x, y, z \geq 0 \end{cases}$,

i.e. $f(x,y) = xyz (120-x-y)$ on $\Delta$.

$f$ is $\Omega$ on $\Delta$.

So consider $\delta = \nabla f = \nabla (120xy - x^2 - y^2)$

$\Rightarrow (0,0) = (120y - 2x - y, 120x - 2x - x)$

$\Rightarrow y(120 - 2x - y) = 0 = x(120 - 2y - x)$.

$\Rightarrow 120 - 2x - y = 0 = 120 - 2y - x$

(don't want $x$ or $y = 0$)  $\Rightarrow y = x$

$\Rightarrow 120 - 3x = 0 \Rightarrow x = 40$

$\Rightarrow y = 40 \Rightarrow z = 40$

$\Rightarrow \text{max} = 40^3 = 64,000$
Ex 3: Find 3 positive numbers that add up to 120 such that their product is a maximum.

What are the set of functions f?

Want to maximize \( xy^2 \) on \( \{x+y+z=120, x,y,z \geq 0\} \).

i.e. \( f(x,y) = xy(120-x-y) \) on \( \Delta \).

\( f \) is \( \leq \) on \( \Delta \).

So consider \( \delta = \nabla f = \nabla (120xy-x^2-y^2) \)

\( = (0,0) = (120 - 2xy - y^2, 120 - 2xy - x^2) \)

\( \Rightarrow y(120 - 2x - y) = 0 = x(120 - 2y - x) \).

\( \Rightarrow 120 - 2x - y = 0 = 120 - 2y - x \)

(\( \text{don't want } x \text{ or } y = 0 \)) \( \Rightarrow y = x \)

\( \Rightarrow 120 - 3x = 0 \Rightarrow x = 40 \)

\( \Rightarrow y = 40 \Rightarrow z = 40 \)

\( \Rightarrow \text{max} = 40^3 = 64,000 \)

Ex 4: Find the minimum distance between the origin & the surface \( z^2 = x^2 + y^4 \).
Ex 3: Find 3 positive numbers that add up to 120 such that their product is a maximum. What are the set $\mathcal{D}$ and function $f$? Want to maximize $xy^2$ on $\begin{cases} x + y + z = 120 \\ x, y, z \geq 0 \end{cases}$, i.e., $f(x,y) = xy(120-x-y)$ on $\mathcal{D}$.

So consider $\mathbf{\nabla} f = \mathbf{\nabla} (120xy-x^2-y^2)$

$\implies (0,0) = (120y - 2xy - y^2, 120x - 2xy - x^2)$

$\implies y(120 - 2x - y) = 0 = x(120 - 2y - x)$.

$\implies 120 - 2x - y = 0 = 120 - 2y - x$ (don't want $x$ or $y = 0$) $\implies y = x$.

$\implies 120 - 3x = 0 \implies x = 40$

$\implies y = 40 \implies z = 40$

$\implies \text{max} = 40^3 = 64,000$

Ex 4: Find the minimum distance between the origin $O$ and the surface $z^2 = x^2 + y^2 + 4$.

Let $P = (x, y, z)$ be any point on the surface. $\text{dist}(P, O) = \sqrt{x^2 + y^2 + z^2}$ is easier to work with. So consider $f(x,y) = x^2 + y^2 + (z^2 + 4)$.
Ex 3 | Find 3 positive numbers that add up to 120 & such that their product is a maximum.

What are the set & & function $f$?

Want to maximize $xy^2$ on $\begin{cases} x+y+z=120 \\ x, y, z \geq 0 \end{cases}$, i.e. $f(x, y) = xy(120-x-y)$ on

\[
\begin{array}{c}
(0, 0) \\
(0, 120) \\
(120, 0)
\end{array}
\]

$f$ is $\leq$ on $\mathbb{R}^3$.

So consider $\delta = \nabla f = \nabla (120xy - x^2 - y^2)

\Rightarrow (0, 0) = (120y - 2xy, 120x - 2xy - x)

\Rightarrow (0, 0) = (0, 0)

\Rightarrow 120 - 2x - y = 0 = 120 - 2y - x

(don't want $x = y = 0$) \Rightarrow y = x

\Rightarrow 120 - 3x = 0 \Rightarrow x = 40

\Rightarrow y = 40 \Rightarrow z = 40

\Rightarrow \max = 40^3 = 64,000$

Ex 4 | Find the minimum distance between the origin & the surface $z^2 = x^2 + y^4$.

Let $P = (x, y, z)$ be any point on the surface.

(Dist ($P, O$))$^{2} = x^2 + y^2 + z^2$ is easier to work with.

So consider $f(x, y) = x^2 + y^2 + (x^2 y^4 + 4)$. Start by finding the critical points:

\[
\begin{align*}
\hat{0} &= \nabla f = (2x + 2xy, 2y + x^2) \\
\Rightarrow y &= -\frac{x^2}{2} \\
\Rightarrow 0 &= 2x + 2xy = 2x - x^2 \\
\Rightarrow x &= 0, \pm \sqrt{2}
\end{align*}
\]

$\Rightarrow (x, y) : (0, 0), (\sqrt{2}, -1), (-\sqrt{2}, -1)$. 
Ex 3  Find 3 positive numbers that add up to 120 & such that their product is a maximum.

What are the set $\mathcal{S}$ & function $f$?
Want to maximize $xy^2$ on $x + y + z = 120$
\[ x, y, z \geq 0 \]
\[ f(x, y) = xy(120-x-y) \]
\[ f(0, 0) = 0 \]
\[ f(120, 0) = 0 \]
\[ f(0, 120) = 0 \]

So consider $\Delta = \nabla f = \nabla (120xy - x^2 - y^2)$
\[ = (120y - 2xy - x^2, 120y - 2xy - y^2) \]
\[ = y(120 - 2x - y) = 0 \]
\[ = x(120 - 2y - x) \]
\[ 120 - 2x - y = 0 \]
\[ 120 - 2y - x = 0 \]
\[ (don't\, want\, x = y = 0) \]
\[ y = x \]
\[ 120 - 3x = 0 \]
\[ x = 40 \]
\[ y = 40 \]
\[ z = 40 \]
\[ \therefore \text{max} = 40^3 = 64,000 \]

Ex 4  Find the minimum distance between the origin $\mathcal{O}$ & the surface $z^2 = x^2 + y^2 + 4$.

Let $P = (x, y, z)$ be any point on the surface.
\[ (\text{dist} (P, O))^2 = x^2 + y^2 + z^2 \]

is easier to work with.

So consider $f(x, y) = x^2 + y^2 + (x^2 + y^2 + 4)$.

Start by finding the critical points:
\[ \nabla f = (2x + 2xy, 2y + x^2) \]
\[ \Rightarrow y = -x^2 \]
\[ \Rightarrow 0 = 2x + 2xy = 2(x + y) \]
\[ \Rightarrow x = 0, \pm \sqrt{2} \]
\[ y = 0, \pm \sqrt{2} \]
\[ \Rightarrow (x, y) = (0, 0), (\sqrt{2}, -1), (-\sqrt{2}, -1). \]

Since $x, y$ don't live in a bounded set $\mathcal{S}$ this time, we take $\mathcal{S}$ to be an arbitrarily large dish about $\mathcal{O}$ and note that $f$ is at least the radius of this dish on the boundary. So the minimum must occur on the interior.

Evaluating $f(0, 0) = 4$ we see that $f(\sqrt{2}, -1) = 5$ then the min dist. = $\sqrt{9}$ $f(-\sqrt{2}, -1) = 5$ $\Rightarrow 2$. 
The last 2 examples are constrained extremum problems, which in general can be quite difficult; in particular, sometimes you can't use the constraint to solve for one variable in terms of the others.
The last 2 examples are constrained extremum problems, which in general can be quite difficult; in particular, sometimes you can't use the constraint to solve for one variable in terms of the others.

Fortunately, there is a better approach. Consider first the case where we want to optimize $f(x,y)$ subject to $g(x,y) = 0$. 

The diagram illustrates level curves of $f$ and the constraint curve $g = 0$. The gradient of $f$ is shown at a point on the constraint curve, indicating the direction of steepest ascent or descent along the constraint.
The last 2 examples are constrained extremum problems, which in general can be quite difficult: in particular, sometimes you can't use the constraint to solve for one variable in terms of the others.

Fortunately, there is a better approach. Consider first the case where we want to optimize \( f(x,y) \) subject to \( g(x,y) = 0 \).

We shall take it to be geometrically evident that the level curve \( f = k \) with greatest possible \( k \) intersecting the constraint curve is tangent to it at the intersection point(s). Hence, their normal vectors are parallel, and

\[ \nabla f \parallel \nabla g \]

at \( (x_0,y_0) \) on \( g = 0 \) where \( f \) is maximized (or minimized).
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Here, their normal vectors are parallel, and \( \nabla f \parallel \nabla g \) at \((x_0, y_0)\) on \( g = 0 \) where \( f \) is maximized (or minimized).

A little less heuristically, if \( \hat{f}(t) = \hat{x}(t), \hat{y}(t) \) parametrizes \( g(x,y) = 0 \), then the function \( f \) is maximized on \( g = 0 \) where

\[
0 = \frac{d}{dt} f(\hat{f}(t)) = \nabla f(\hat{f}(t)) \cdot \hat{r}'(t)
\]

for \( t = \) some \( t_e \); and then \( \nabla f(\hat{f}(t_e)) \perp \hat{r}'(t_e) \).
The last 2 examples are constrained extremum problems, which in general can be quite difficult; in particular, sometimes you can't use the constraint to solve for one variable in terms of the others.

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Here, their normal vectors are parallel, and \( \nabla f = \nabla g \) at \((x_0, y_0)\) on \( g = 0 \) where \( f \) is maximized (or minimized).

A little less heuristically, if \( \hat{f}(x) = \hat{g}(x), y(x) \) parametrizes \( g(x, y) = 0 \), then the function \( f \) is maximized on \( g = 0 \) where

\[
0 = \frac{d}{dx} f(\hat{f}(x)) = \nabla f(\hat{f}(x)) \cdot \hat{f}'(x)
\]

for \( t = \text{some } t_0 \); and then \( \nabla f(\hat{f}(x_0)) \perp \hat{f}'(x_0) \).

But \( \nabla g(\hat{f}(x_0)) \perp \nabla f(\hat{f}(x_0)) \) as well (since gradient is normal to \( g = 0 \), and \( \hat{f}'(x_0) \) is tangent to \( f \)) so

\[
\nabla f(\hat{f}(x_0)) \parallel \nabla g(\hat{f}(x_0)).
\]
This approach to solving constrained extremum problems generalizes to 3, 4, etc. variables.
For 3 variables, here's the step-by-step:

**Step 1:** Make sure the constraint is in the form \( g(x, y, z) = 0 \). [e.g. \( x^2 + y^2 = z^2 \) is not]
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**Step 1:** Make sure the constraint is in the form \( g(x, y, z) = 0 \). [e.g. \( x^2 + y^2 = z^2 \) is not]

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**Ex 5** Find the greatest area a rectangle can have with diagonal of length 2.
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---

**Ex 5** Find the greatest area a rectangle can have with diagonal of length 2.

1) \( f(x, y) = xy, \quad g(x, y) = x^2 + y^2 - 4 \) (why?)

2) \( \nabla f = \lambda \nabla g \rightarrow (y, x) = \lambda (2x, 2y) \)
Ex 5 | Find the greatest area a rectangle can have with diagonal of length 2.

1) \( f(x,y) = xy, \quad g(x,y) = x^2+y^2-4 \) (why?)
2) \( \nabla f = \lambda \nabla g \rightarrow (x,y) = \lambda (2x, 2y) \)
3) \( y = 2\lambda x, \quad x = 2\lambda y, \quad x^2 + y^2 = 4 \)

\[
\begin{align*}
x^2 y &= 2\lambda^2 x^2 \\
y^2 x &= 2\lambda^2 y^2 \\
x^2 + y^2 &= 4
\end{align*}
\]

\[
\begin{align*}
x^2 &= 4 \\
x &= \pm 2
\end{align*}
\]

\[
\begin{align*}
y &= \pm \sqrt{2} \\
\end{align*}
\]

or \( \lambda = 0 \) impossible; would give \( x = 0 \), \( xy \neq 0 \)

4) \( f(2, \sqrt{2}) = 2 = A_{\text{max}} \).
This approach to solving constrained extremum problems generalizes to 3, 4, etc. variables. For 3 variables, here's the step-by-step:

**Step 1:** Make sure the constraint is in the form $g(x,y,z) = 0$. [e.g. $x^2 + y^2 = z^2$ is not]

**Step 2:** Set $\nabla f = \lambda \nabla g$

**Step 3:** Solve the resulting set of equations (3 from Step 2, 1 from Step 1: $g = 0$) to find $(x, y, z)$.

**Step 4:** Evaluate $f$ at each resulting solution $(x, y, z)$ to see when it is really biggest/smallest.

**Ex 5** Find the greatest area a rectangle can have with diagonal of length 2.

1) $f(x,y) = xy$, $g(x,y) = x^2 + y^2 - 4$ (why?)
2) $\nabla f = \lambda \nabla g \rightarrow (y, x) = \lambda (2x, 2y)$
3) $y = 2\lambda x$, $x = 2\lambda y$, $x^2 + y^2 = 4$
   \[ \begin{align*}
   y &= 2\lambda x \\
   x &= 2\lambda y \\
   x^2 + y^2 &= 4
   \end{align*} \]
   \[ \begin{align*}
   x &= 2 \Rightarrow x^2 = 4 \\
   y &= \sqrt{4} \\
   \lambda &= 0 \quad \text{impossible: would give } x = 0 \text{ or } y = 0 \Rightarrow f \neq 0
   \end{align*} \]
4) $f(\sqrt{2}, \sqrt{2}) = 2 = A_{\text{max}}$

As you know, the method we've been describing is that of Lagrange multipliers. We will continue with this next time.