Lecture 33

Lagrange Multipliers
Ex 1: Find the max/min values of 
\[ f(x, y) = y^2 - x^2 \] on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)
Ex 1 | Find the max/min values of

\[ f(x,y) = y^2 - x^2 \]

on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lec. 38,

(1) \( g(x,y) = \frac{x^2}{4} + y^2 - 1 \)
Ex 1  Find the max/min values of 

\[ f(x, y) = y^2 - x^2 \]  on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of lect. 38,

1. \( g(x, y) = \frac{x^2}{4} + y^2 - 1 \)
2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{x}{4}, 2y) \)
Ex 1 | Find the max/min values of 
\( f(x, y) = y^2 - x^2 \) on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lect. 32,

1. \( g(x, y) = \frac{x^2}{4} + y^2 - 1 \)
2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{x}{2}, 2y) \)

\[ \Rightarrow \begin{align*}
(A) & \quad -2x = \frac{\lambda x}{2} \\
(B) & \quad 2y = 2\lambda y \\
(C) & \quad \frac{x^2}{4} + y^2 = 1 \quad (or \quad x^2 + 4y^2 = 4)
\end{align*} \]
Ex 1 | Find the max/min values of
\[ f(x, y) = y^2 - x^2 \] on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lec. 38,

1. \( g(x, y) = \frac{x^2}{4} + y^2 - 1 \)

2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{1}{2}x, 2y) \)

   \( \Rightarrow \)
   \( (A) \) \(-2x = \frac{x}{2} \), \( (B) \) \(2y = 2\lambda y\), and
   \( (C) \) \(\frac{x^2}{4} + y^2 = 1 \) (or \( x^2 + 4y^2 = 4 \))

3. \( (A) \Rightarrow -4x = \lambda x \Rightarrow (i) \lambda = -4 \) or (ii) \( x = 0 \).
Ex 1 | Find the max/min values of 
\[ f(x, y) = y^2 - x^2 \] on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lect.38,

1. \( g(x, y) = \frac{x^2}{4} + y^2 - 1 \)

2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\hat{x}, 2y) \)
   \[ \Rightarrow (A) \quad -2x = \frac{\lambda x}{2}, \quad (B) \quad 2y = 2\lambda y, \quad \text{and} \]
   \[ (C) \quad \frac{x^2}{4} + y^2 = 1 \quad (\text{or} \quad x^2 + 4y^2 = 4) \]

3. \( (A) \Rightarrow -4x = \lambda x \Rightarrow (i) \quad \lambda = -4 \quad \text{or} \quad (ii) x = 0. \)

   (i) : \( (B) \) becomes \( 2y = -8y \Rightarrow y = 0, \) whence
   \[ (C) \text{ yields } x = \pm 2 \quad \Rightarrow (\pm 2, 0) \]
Ex 1: Find the max/min values of
\[ f(x, y) = y^2 - x^2 \] on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lect. 38,

1. \( g(x, y) = \frac{x^2}{4} + y^2 - 1 \)
2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \)

\[ \Rightarrow (A) \quad -2x = \frac{\lambda x}{2}, \quad (B) \quad 2y = 2\lambda y, \quad \text{and} \]
\[ (C) \quad \frac{x^2}{4} + y^2 = 1 \quad \text{(or} \quad x^2 + 4y^2 = 4) \]

3. \( (A) \Rightarrow -4x = \lambda x \Rightarrow (i) \lambda = -4 \quad \text{or} \quad (ii) x = 0. \)

(ii): (B) becomes \( 2y = -8y \Rightarrow y = 0 \), whence
\[ (C) \quad \text{yields} \quad x = \pm 2 \quad \Rightarrow \quad (\pm 2, 0) \]

(iii): (C) becomes \( 4y^2 = 4 \Rightarrow y = \pm 1 \quad \Rightarrow \quad (0, \pm 1) \)
Ex 1: Find the max/min values of 
\[ f(x, y) = y^2 - x^2 \] on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lect. 38,

1. \[ g(x, y) = \frac{x^2}{4} + y^2 - 1 \]

2. \[ \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{x}{2}, 2y) \]
   \[ \Rightarrow (A) -2x = \frac{\lambda x}{2}, \quad (B) 2y = 2\lambda y, \text{ and} \]
   \[ (C) \frac{x^2}{4} + y^2 = 1 \quad (\text{or} \ x^2 + 4y^2 = 4) \]

3. \( (A) \Rightarrow -4x = \lambda x \Rightarrow (i) \lambda = -4 \text{ or (ii) } x = 0. \)
   \( (ii): (B) \text{ becomes } 2y = -8y \Rightarrow y = 0, \text{ whence} \)
   \[ (C) \text{ yields } x = \pm 2 \quad \Rightarrow (\pm 2, 0) \]
   \( (iii): (C) \text{ becomes } 4y^2 = 4 \Rightarrow y = \pm 1 \quad \Rightarrow (0, \pm 1) \)

4. Plug into \( f \):
   \[ f(2, 0) = -4 = f(-2, 0) \quad \text{MIN} \]
   \[ f(0, 1) = 1 = f(0, -1) \quad \text{MAX} \]
Ex 1: Find the max/min values of $f(x,y) = y^2 - x^2$ on the ellipse $\frac{x^2}{4} + y^2 = 1$

Following the steps from the end of lect. 38,

1. $g(x,y) = \frac{x^2}{4} + y^2 - 1$

2. $\nabla f = \lambda \nabla g \implies (-2x, 2y) = \lambda (2x, 2y)$

   $\Rightarrow (A) \quad -2x = \frac{2x}{\lambda}$, (B) $2y = 2\lambda y$, and

   (C) $\frac{x^2}{4} + y^2 = 1$ (or $x^2 + 4y^2 = 4$)

3. (A) $\Rightarrow -4x = \lambda x \implies (i) \lambda = -4$ or (ii) $x = 0$.

   (i): (B) becomes $2y = -8y \implies y = 0$, whence

   (C) yields $x = \pm 2 \implies (\pm 2, 0)$

   (ii): (C) becomes $4y^2 = 1 \implies y = \pm 1 \implies (0, \pm 1)$

4. Plug into $f$:

   $f(2,0) = -4 = f(-2,0)$ MIN

   $f(0,1) = 1 = f(0, -1)$ MAX

Ex 2: Optimize $f(x,y) = e^{-xy}$ on the elliptical disk $x^2 + 4y^2 \leq 1$. [Diagram]
Ex 1  Find the max/min values of $f(x,y) = y^2 - x^2$ on the ellipse $\frac{x^2}{4} + y^2 = 1$

Following the steps from the end of Lec. 32,

1. $g(x,y) = \frac{x^2}{4} + y^2 - 1$

2. $\nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{\lambda x}{2}, 2y)$

   $\Rightarrow (A) \ -2x = \frac{\lambda x}{2}, \ (B) \ 2y = 2\lambda y, \ and$

   (c) $\frac{x^2}{4} + y^2 = 1$ (or $x^2 + 4y^2 = 4$)

3. (A) $\Rightarrow -4x = \lambda x \Rightarrow (i) \ \lambda = -4 \ or \ (ii) \ x = 0$

   (i): (B) becomes $2y = -8y \Rightarrow y = 0$, whence

   (c) yields $x = \pm 2 \rightarrow (\pm 2, 0)$

   (ii) (c) becomes $4y^2 = 4 \Rightarrow y = \pm 1 \rightarrow (0, \pm 1)$

4. Plug into $f$:

   $f(2, 0) = -4 = f(-2, 0) \text{ MIN}$

   $f(0, 1) = 1 = f(0, -1) \text{ MAX}$

Ex 2  Optimize $f(x,y) = e^{-xy}$ on the elliptical disk $x^2 + 4y^2 \leq 1$.

First let's find the critical (stationary) points in the interior: $\nabla f = \begin{pmatrix} -ye^{-xy} \\ -xe^{-xy} \end{pmatrix} \Rightarrow (x, y) = (0, 0)$ at which $f(0, 0) = 1$. 
Ex 1] Find the max/min values of $f(x,y) = y^2 - x^2$ on the ellipse $\frac{x^2}{4} + y^2 = 1$.

Following the steps from the end of Lect. 33:

1. $g(x,y) = \frac{x^2}{4} + y^2 - 1$
2. $\nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{dx}{dx}, 2y)$
   $\Rightarrow (A) -2x = \frac{\lambda x}{2}$, (B) $2y = 2\lambda y$, and
   (C) $\frac{x^2}{4} + y^2 = 1$ (or $x^2 + 4y^2 = 4$)
3. (A) $\Rightarrow -4x = \lambda x \Rightarrow (i) \lambda = -4$ or (ii) $x = 0$.
   (i): (B) becomes $2y = -8y \Rightarrow y = 0$, whence
   (C) yields $x = \pm 2 \rightarrow (\pm 2, 0)$
   (ii): (C) becomes $4y^2 = 4 \Rightarrow y = \pm 1 \rightarrow (0, \pm 1)$
4. Plug into $f$:
   $f(2,0) = -4 = f(-2,0)$ MIN
   $f(0,1) = 1 = f(0,-1)$ MAX

Ex 2] Optimize $f(x,y) = e^{-xy}$ on the elliptical disk $x^2 + 4y^2 \leq 1$.

First let’s find the critical (stationary) points in the interior: $\hat{0} = \nabla f = (-ye^{-x^2}, -xe^{-y^2}) \Rightarrow (x,y) = (0,0)$ at which $f(0,0) = 1$.

Turning to the boundary, to find max/min there we use Lagrange with $g(x,y) = x^2 + 4y^2 - 1.$
Ex 1] Find the max/min values of 

\[ f(x, y) = y^2 - x^2 \] 
on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lect. 32,

1. \( g(x, y) = \frac{x^2}{4} + y^2 - 1 \)

2. \( \nabla f = \lambda \nabla g \) \( \Rightarrow (2x, 2y) = \lambda (\frac{x}{2}, y) \)

\[ \Rightarrow (A) \quad -2x = \frac{x}{2}, \quad (B) \quad 2y = 2\lambda y, \quad \text{and} \]
\[ (C) \quad \frac{x^2}{4} + y^2 = 1 \quad (\text{or } x^2 + 4y^2 = 4) \]

3. \( \Rightarrow -4x = \lambda x \) \( \Rightarrow \) (i) \( \lambda = -4 \) or (ii) \( x = 0 \).

(i): \( B \) becomes \( 2y = -8y \) \( \Rightarrow y = 0 \), whence

\( \text{(C) yields } x = \pm 2 \) \( \Rightarrow \) \( (\pm 2, 0) \)

(ii): \( C \) becomes \( 4y^2 = 1 \) \( \Rightarrow y = \pm 1 \) \( \Rightarrow \) \( (0, \pm 1) \)

4. Plug into \( f \):

\[ f(2, 0) = -4 = f(-2, 0) \quad \text{MIN} \]

\[ f(0, 1) = 1 = f(0, -1) \quad \text{MAX} \]

Ex 2] Optimize \( f(x, y) = e^{-xy} \) on the elliptical disk \( x^2 + 4y^2 \leq 1 \).

First let's find the critical (stationary) points in the interior: \( \hat{0} = \nabla f = (-ye^{-xy}, -xe^{-xy}) \Rightarrow (x, y) = (0, 0) \) at which \( f(0, 0) = 1 \).

Turning to the boundary, to find max/min, then we use Lagrange with \( g(x, y) = x^2 + 4y^2 - 1 \).

\( \nabla f = \lambda \nabla g \) is \( (-ye^{-xy}, -xe^{-xy}) = \lambda (2x, 8y) \).

- \( -ye^{-xy} = 2\lambda x \) \( \Rightarrow \) \( -xye^{-xy} = 2\lambda x^2 \)
- \( -xe^{-xy} = 8\lambda y \) \( \Rightarrow \) \( -xye^{-xy} = 8\lambda y^2 \)
**Ex 1** Find the max/min values of \( f(x, y) = y^2 - x^2 \) on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lect. 32,

1. \( g(x, y) = \frac{x^2}{4} + y^2 - 1 \)
2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{x}{2}, 2y) \)
   \[ \Rightarrow \begin{align*}
   (A) \quad & -2x = \frac{x}{2} \quad \text{and} \\
   & (B) \quad 2y = 2\lambda y, \quad \text{and} \\
   & (C) \quad \frac{x^2}{4} + y^2 = 1 \quad \text{(or} \quad x^2 + 4y^2 = 4 \text{)}
   \end{align*} \]
3. \( \begin{align*}
   (A) \Rightarrow -4x = \lambda x & \Rightarrow (i) \lambda = -4 \quad \text{or} \quad (ii) x = 0.
   \end{align*} \)
   
   (i): \( (B) \) becomes \( 2y = -8y \Rightarrow y = 0 \), whence
   
   \( c1 \) yields \( x = \pm 2 \quad \Rightarrow (\pm 2, 0) \)

   (ii): \( (C) \) becomes \( 4y^2 = 4 \Rightarrow y = \pm 1 \quad \Rightarrow (0, \pm 1) \)

4. Plug into \( f \):
   
   \( f(2, 0) = -4 = f(-2, 0) \) \quad \text{MIN}
   
   \( f(0, 1) = 1 = f(0, -1) \) \quad \text{MAX}

**Ex 2** Optimize \( f(x, y) = e^{-xy} \) on the elliptical disk \( x^2 + 4y^2 \leq 1 \).

First let's find the critical (stationary) points in the interior: \( \nabla f = (-ye^{-xy}, -xe^{-xy}) \Rightarrow \nabla f = (0, 0) \) at which \( f(0, 0) = 1 \).

Turning to the boundary, to find max/min there we use Lagrange with \( g(x, y) = x^2 + 4y^2 - 1 \).

\( \nabla f = \lambda \nabla g \) is \( (-ye^{-xy}, -xe^{-xy}) = \lambda (2x, 8y) \).

- \(-ye^{-xy} = 2\lambda x \Rightarrow \lambda = \frac{-y}{2x} \cdot \text{e}^{-xy} = 2\lambda x^2 \)
- \(-xe^{-xy} = 8\lambda y \Rightarrow \lambda = \frac{-x}{8y} \cdot \text{e}^{-xy} = 8\lambda y^2 \)

\( \Rightarrow 2\lambda x^2 = 8\lambda y^2 \Rightarrow \lambda = 0 \) or \( 2x^2 = 8y^2 \)
**Ex 1** Find the max/min values of $f(x,y) = y^2 - x^2$ on the ellipse $\frac{x^2}{4} + y^2 = 1$

Following the steps from the end of Lect. 28:

1. $g(x,y) = \frac{x^2}{4} + y^2 - 1$
2. $\nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\pm x, 2y)$
   
   $\Rightarrow (A) -2x = \pm \frac{x}{2}, (B) 2y = 2\lambda y,$ and
   
   (c) $\frac{x^2}{4} + y^2 = 1$ (or $x^2 + 4y^2 = 4$)
3. (A) $\Rightarrow -4x = \lambda x \Rightarrow (i) \lambda = -4$ or (ii) $x = 0$.

   (i): (B) becomes $2y = -8y \Rightarrow y = 0$ , whence
   
   (c) yields $x = \pm 2 \Rightarrow (\pm 2, 0)$

   (ii): (c) becomes $4y^2 = 4 \Rightarrow y = \pm 1 \Rightarrow (0, \pm 1)$

4. Plug into $f$:
   
   $f(2,0) = -4 = f(-2, 0)$ MIN
   
   $f(0,1) = 1 = f(0, -1)$ MAX

**Ex 2** Optimize $f(x,y) = e^{-xy}$ on the elliptical disk $x^2 + 4y^2 \leq 1$.

First let's find the critical (stationary) points in the interior: $\hat{0} = \nabla f = (-ye^{-xy}, -xe^{-xy}) \Rightarrow (x,y) = (0, 0)$ at which $f(0,0) = 1$.

Turning to the boundary, to find max/min there we use Lagrange with $g(x,y) = x^2 + 4y^2 - 1$.

$\nabla f = \lambda \nabla g$ is $(-ye^{-xy}, -xe^{-xy}) = \lambda (2x, 8y)$.

- $-ye^{-xy} = 2\lambda x \Rightarrow -xye^{-xy} = 2\lambda x^2$
- $-xe^{-xy} = 8\lambda y \Rightarrow -xye^{-xy} = 8\lambda y^2$

$\Rightarrow 2\lambda x^2 = 8\lambda y^2 \Rightarrow x = 0$ or $2x^2 = 8y^2$ 

This would imply $x = 0 = y$ (not on constraint $g = 0$ !)
Ex 1 \quad \text{Find the max/min values of } f(x,y) = y^2 - x^2 \text{ on the ellipse } \frac{x^2}{4} + y^2 = 1.

Following the steps from the end of Lecture 3,

1. \( g(x,y) = \frac{x^2}{4} + y^2 - 1 \)
2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\frac{x}{2}, 2y) \)

\[ \Rightarrow \begin{cases} (A) -2x = \frac{x}{2} & \text{(b)} 2y = 2\lambda y, \text{ and} \\ (C) \frac{x^2}{4} + y^2 = 1 \text{ (or } x^2 + 4y^2 = 4) \end{cases} \]

3. (A) \( -4x = \lambda x \Rightarrow (i) \lambda = -4 \text{ or } (ii) x = 0. \)

   (i): (B) becomes \( 2y = -8y \Rightarrow y = 0, \) whence
   (C) yields \( x = \pm 2 \quad \Rightarrow (\pm 2, 0) \)

   (ii): (C) becomes \( 4y^2 = 1 \Rightarrow y = \pm 1 \Rightarrow (0, \pm 1) \)

4. Plug into \( f: \)

\[ \begin{align*}
 f(2,0) &= -4 = f(-2,0) \quad \text{MIN} \\
 f(0,1) &= 1 = f(0,-1) \quad \text{MAX}
\end{align*} \]

Ex 2 \quad \text{Optimize } f(x,y) = e^{-xy} \text{ on the elliptical disk } x^2 + 4y^2 \leq 1.

First let's find the critical (stationary) points in the interior: \( \hat{0} = \nabla f = (-ye^{-xy}, -xe^{-xy}) \Rightarrow (x,y) = (0,0) \) at which \( f(0,0) = 1. \)

Turning to the boundary, to find max/min, then we use Lagrange with \( g(x,y) = x^2 + 4y^2 - 1. \) \n\( \nabla f = \lambda \nabla g \) is \( (-ye^{-xy}, -xe^{-xy}) = \lambda (2x, 8y). \)

- \( -ye^{-xy} = 2\lambda x \quad \Rightarrow -xye^{-xy} = 2\lambda x^2 \)
- \( -xe^{-xy} = 8\lambda y \quad \Rightarrow -xye^{-xy} = 8\lambda y^2 \)

\[ \Rightarrow 2\lambda x^2 = 8\lambda y^2 \Rightarrow x = \pm 2y \quad \Rightarrow 8y^2 = 1 \Rightarrow y = \pm \frac{1}{2\sqrt{2}}. \]
Ex 1] Find the max/min values of 
\[ f(x,y) = y^2 - x^2 \]
on the ellipse \( \frac{x^2}{4} + y^2 = 1 \)

Following the steps from the end of Lect. 38:

1. \( g(x,y) = \frac{x^2}{4} + y^2 - 1 \)
2. \( \nabla f = \lambda \nabla g \Rightarrow (-2x, 2y) = \lambda (\pm x, 2y) \)
   \[ \Rightarrow (A) \ -2x = \frac{x}{2}, \ (B) \ 2y = 2\lambda y, \ \text{and} \]
   \[ (C) \frac{x^2}{4} + y^2 = 1 \] (or \( x^2 + 4y^2 = 4 \))
3. (A) \( -4x = \lambda x \Rightarrow (i) \ \lambda = -4 \) or (ii) \( x = 0 \).
   (i): (B) becomes \( 2y = -8y \Rightarrow y = 0 \), whence
   \( (C) \) yields \( x = \pm 2 \) \( \Rightarrow (\pm 2, 0) \)
   (ii): (C) becomes \( 4y^2 = 1 \) \( \Rightarrow y = \pm 1 \) \( \Rightarrow (0, \pm 1) \)
4. Plug into \( f \):
   \[ f(2,0) = -4 = f(-2,0) \quad \text{MIN} \]
   \[ f(0,1) = f(0,-1) = 1 \quad \text{MAX} \]

Ex 2] Optimize \( f(x,y) = e^{-xy} \) on the elliptical disk \( x^2 + 4y^2 \leq 1 \).

First let's find the critical (stationary) points in the interior: \( \hat{0} = \nabla f = (-ye^{-xy}, -xe^{-xy}) \Rightarrow (x,y) = (0,0) \) at which \( f(0,0) = 1 \).

Turning to the boundary, to find max/min there we use Lagrange with \( g(x,y) = x^2 + 4y^2 - 1 \).
\[ \nabla f = \lambda \nabla g \] is \( (-ye^{-xy}, -xe^{-xy}) = \lambda (2x, 8y) \).
- \( -ye^{-xy} = 2\lambda x \quad \Rightarrow -xe^{-xy} = 2\lambda x \)
- \( -xe^{-xy} = 8\lambda y \quad \Rightarrow -ye^{-xy} = 8\lambda y \)
\[ \Rightarrow 2\lambda x^2 = 8\lambda y^2 \quad \Rightarrow 2x^2 = 8y^2 \]
\[ \Rightarrow x = \pm 2y \quad \Rightarrow 8y^2 = 1 \Rightarrow y = \pm \frac{1}{2} \sqrt{2} \]
\( \text{use} \) \( x^2 + 4y^2 = 1 \) \( \Rightarrow x = \pm \frac{1}{2} \sqrt{2} \).
Now compare \( f\left(\frac{1}{2\sqrt{2}}, \frac{1}{2}\right) = f\left(\frac{1}{2\sqrt{2}}, -\frac{1}{2}\right) = e^{1/4} < 1 \quad \text{MIN} \),
\( f\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2}\right) = f\left(\frac{1}{2\sqrt{2}}, -\frac{1}{2}\right) = e^{1/4} > 1 \), and \( f(0,0) = 1 \).
General ansatz for Lagrange multipliers:
General ansatz for Lagrange multipliers:

To find all candidates for extrema of $f(x_1, \ldots, x_n)$ subject to $m$ constraints

$$g_1(x) = 0, \ g_2(x) = 0, \ldots, \ g_m(x) = 0,$$
General ansatz for Lagrange multipliers:

To find all candidates for extrema of
\( f(x_1, \ldots, x_n) \) subject to \( m \) constraints
\[
g_1(x) = 0, \quad g_2(x) = 0, \ldots, \quad g_m(x) = 0,
\]
we need to solve the system
\[
\begin{align*}
g_1(x) &= 0 \\
g_2(x) &= 0 \\
g_m(x) &= 0 \\
\nabla f(x) &= \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x)
\end{align*}
\]
General ansatz for Lagrange multipliers:

To find all candidates for extrema of

\( f(x_1, \ldots, x_n) \) subject to \( m \) constraints

\( g_1(x) = 0, \ g_2(x) = 0, \ldots, \ g_m(x) = 0, \)

we need to solve the system

\[
\begin{align*}
\nabla f(x) &= \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x) \\
\nabla g_1(x) &= 0 \\
\nabla g_2(x) &= 0 \\
\vdots & \\
\nabla g_m(x) &= 0
\end{align*}
\]

(this last equation is really \( n \) equations)

\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_1} \\
\vdots & \\
\frac{\partial f}{\partial x_n} &= \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_n}
\end{align*}
\]
General ansatz for Lagrange multipliers:

To find all candidates for extrema of

$$f(x_1, ..., x_n)$$ subject to $$m$$ constraints

$$g_1(x) = 0, g_2(x) = 0, ..., g_m(x) = 0,$$

we need to solve the system

$$\begin{cases} 
    g_1(x) = 0 \\
    g_2(x) = 0 \\
    g_m(x) = 0 \\
    \n    \n\end{cases}$$

$$\nabla f(x) = \lambda_1 \nabla g_1(x) + ... + \lambda_m \nabla g_m(x)$$

This last equation is really $$m$$ equations

$$\begin{cases} 
    \frac{\partial f}{\partial x_1} = \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_1} \\
    \vdots \\
    \frac{\partial f}{\partial x_n} = \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_n} \\
\end{cases}$$

So altogether we have $$n+m$$ equations in $$n+m$$ variables. One expects the solution set to be a finite set of points, which will include our desired max/min.
General ansatz for Lagrange multipliers:

To find all candidates for extrema of $f(x_1, \ldots, x_n)$ subject to $m$ constraints

$$g_1(x) = 0, \ g_2(x) = 0, \ldots, \ g_m(x) = 0,$$

we need to solve the system

$$\begin{cases}
g_1(x) = 0 \\
g_2(x) = 0 \\
g_m(x) = 0 \\
\nabla f(x) = \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x)
\end{cases}$$

(This last equation is really $n$ equations)

$$\begin{cases}
\frac{\partial f}{\partial x_1} = \sum_{j=1}^m \lambda_j \frac{\partial g_j}{\partial x_1} \\
\vdots \\
\frac{\partial f}{\partial x_n} = \sum_{j=1}^m \lambda_j \frac{\partial g_j}{\partial x_n}
\end{cases}$$

So altogether we have $n+m$ equations in $n+m$ variables. One expects the solution set to be a finite set of points, which will include our desired max/min.
General ansatz for Lagrange multipliers:

To find all candidates for extrema of \( f(x_1, \ldots, x_n) \) subject to \( m \) constraints
\[
g_1(x) = 0, \quad g_2(x) = 0, \quad \ldots, \quad g_m(x) = 0,
\]
we need to solve the system
\[
\begin{align*}
g_1(x) &= 0 \\
g_2(x) &= 0 \\
\vdots & \quad \vdots \\
g_m(x) &= 0 \\
\nabla f(x) &= \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x)
\end{align*}
\]

This last equation is really \( n \) equations
\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= \lambda_1 \frac{\partial g_1}{\partial x_1} \\
\vdots & \quad \vdots \\
\frac{\partial f}{\partial x_n} &= \lambda_m \frac{\partial g_m}{\partial x_n}
\end{align*}
\]

So altogether we have \( n+m \) equations in \( n+m \) variables. One expects the solution set to be a finite set of points, which will include our desired max/min.
General ansatz for Lagrange multipliers:

To find all candidates for extrema of \( f(x_1, \ldots, x_n) \) subject to \( m \) constraints

\[
\begin{align*}
g_1(x) &= 0, \\ g_2(x) &= 0, \\ &\vdots \\ g_m(x) &= 0,
\end{align*}
\]

we need to solve the system

\[
\begin{align*}
g_1(x) &= 0 \\ g_2(x) &= 0 \\ &\vdots \\ g_m(x) &= 0 \\ \nabla f(x) &= \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x)
\end{align*}
\]

(this last equation is really \( m \) equations)

\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= \sum_{j=1}^m \lambda_j \frac{\partial g_j}{\partial x_1} \\ &\vdots \\ \frac{\partial f}{\partial x_n} &= \sum_{j=1}^m \lambda_j \frac{\partial g_j}{\partial x_n}
\end{align*}
\]

So altogether we have \( n+m \) equations in \( n+m \) variables. One expects the solution set to be a finite set of points, which will include our desired max/min.

But why should the method work?

\[\text{IDEA}\]

Say \( n=3, m=2 \) and we are trying to

max/min \( f(x,y,z) \) on

\[ C = \{ g_1=0 \} \cap \{ g_2=0 \}. \]

If \( \vec{r}(c) \) parametrizes \( C \), then \( \vec{r}'(c) \) is tangent to \( C \), hence tangent to \( g_1=0 \) & \( g_2=0 \).
General ansatz for Lagrange multipliers:

To find all candidates for extrema of
\[ f(x_1, \ldots, x_n) \] subject to \( m \) constraints
\[ g_1(x) = 0, \ g_2(x) = 0, \ldots, \ g_m(x) = 0, \]
we need to solve the system
\[
\begin{cases}
    g_1(x) = 0 \\
    \vdots \\
    g_m(x) = 0 \\
    \nabla f(x) = \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x)
\end{cases}
\]

This last equation is really \( n \) equations
\[
\begin{cases}
    \nabla f / \nabla x_1 = \frac{\lambda_1}{\lambda_1} \frac{\partial g_1}{\partial x_1} \\
    \vdots \\
    \nabla f / \nabla x_n = \frac{\lambda_1}{\lambda_m} \frac{\partial g_m}{\partial x_n}
\end{cases}
\]

So altogether we have \( n + m \) equations in \( n + m \) variables. One expects the solution set to be a finite set of points, which will include our desired max/min.

But why should the method work?

\textbf{Idea:}
Say \( n = 3, \ m = 2 \) and we are trying to
\( \text{max/min } f(x, y, z) \) on
\[ C = \{ g_1 = 0 \} \cap \{ g_2 = 0 \}. \]
If \( \vec{r}(t) \) parametrizes \( C \), then \( \vec{r}'(t) \) is
tangent to \( C \), hence tangent to \( g_1 = 0 \) and \( g_2 = 0 \).
Thus \( \vec{r}' \perp \nabla g_1, \ \nabla g_2 \).
General ansatz for Lagrange multipliers:

To find all candidates for extrema of $f(x_1, \ldots, x_n)$ subject to $m$ constraints $g_1(x) = 0, g_2(x) = 0, \ldots, g_m(x) = 0,$ we need to solve the system

$$\begin{cases} g_1(x) = 0 \\ g_2(x) = 0 \\ \vdots \\ g_m(x) = 0 \end{cases} \text{subject to } \nabla f(x) = \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x) \text{.}$$

(this last equation is really $n$ equations)

$$\begin{cases} \frac{\partial f}{\partial x_1} = \lambda_1 \frac{\partial g_1}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} = \lambda_m \frac{\partial g_m}{\partial x_n} \end{cases} \Rightarrow \nabla f \perp \nabla g_1, \ldots, \nabla g_m \text{.}$$

So altogether we have $n+m$ equations in $n+m$ variables. One expects the solution set to be a finite set of points, which will include our desired max/min.

But why should the method work?

**Idea** Say $n = 3$, $m = 2$, and we are trying to max/min $f(x, y, z)$ on $C = \{g_1 = 0\} \cap \{g_2 = 0\}$.

If $\vec{r}(t)$ parameterizes $C$, then $\vec{r}'(t)$ is tangent to $C$, hence tangent to $g_1 = 0$ and $g_2 = 0$.

Thus $\vec{r}' \perp \nabla g_1$ and $\nabla g_2$.

At an extremum of $f$ on $C$ (at $t = t_0$),

$$0 = \frac{d}{dt} f(\vec{r}(t)) = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t_0) \Rightarrow \nabla f \perp \nabla g_1, \nabla g_2 \text{.}$$
General ansatz for Lagrange multipliers:

To find all candidates for extrema of \( f(x_1, ..., x_n) \) subject to \( m \) constraints
\[ g_1(x) = 0, \ g_2(x) = 0, \ldots, \ g_m(x) = 0, \]
we need to solve the system
\[
\begin{align*}
\nabla f(x) &= \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x) \\
\nabla g_1(x) &= 0 \\
\vdots & \\
\nabla g_m(x) &= 0
\end{align*}
\]
\( \Rightarrow \) this last equation is really \( m \) equations
\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= \lambda_1 \frac{\partial g_1}{\partial x_1} \\
\vdots & \\
\frac{\partial f}{\partial x_n} &= \lambda_m \frac{\partial g_m}{\partial x_n}
\end{align*}
\]
So altogether we have \( m \times n \) equations in \( m \times n \) variables. One expects the solution set to be a finite set of points, which will include our desired max/min.

But why should the method work?

**Idea** Say \( n = 3, \ m = 2 \) and we are trying to
\[
\begin{align*}
\text{max/min} \ f(x, y, z) \text{ on} \\
C = \{ g_1 = 0 \} \cap \{ g_2 = 0 \}
\end{align*}
\]
If \( \vec{r}(t) \) parameterizes \( C \), then \( \vec{r}'(t) \) is tangent to \( C \), hence tangent to \( g_1 = 0 \) \& \( g_2 = 0 \).
Thus \( \vec{r}' \perp \nabla g_1 \) \& \( \nabla g_2 \).
At an extremum of \( f \) on \( C \) (at \( \vec{r}_0 = \vec{r}(t_0) \)),
\[
\frac{\partial f}{\partial t}|_{t=t_0} = \frac{\partial f}{\partial \vec{r}}(\vec{r}_0) \cdot \vec{r}'(t_0)
\]
\( \Rightarrow \vec{r}' \perp \nabla f \).
Therefore, if \( \nabla g_1(\vec{r}_0) \) \& \( \nabla g_2(\vec{r}_0) \) span the
subspace \( \perp \vec{r}'(\vec{r}_0) \), we have
\[
\frac{\partial f}{\partial \vec{r}}(\vec{r}_0) = \lambda_1 \nabla g_1(\vec{r}_0) + \lambda_2 \nabla g_2(\vec{r}_0)
\]
for some \( \lambda_1, \lambda_2 \in \mathbb{R} \).
General ansatz for Lagrange multipliers:

To find all candidates for extrema of \( f(x_1, \ldots, x_n) \) subject to \( m \) constraints
\[
\begin{align*}
g_1(x) &= 0, \\
g_2(x) &= 0, \\
&
\vdots \\
g_m(x) &= 0,
\end{align*}
\]
we need to solve the system
\[
\begin{align*}
g_1(x) &= 0, \\
g_2(x) &= 0, \\
&
\vdots \\
g_m(x) &= 0,
\end{align*}
\]
\[
\nabla f(x) = \lambda_1 \nabla g_1(x) + \cdots + \lambda_m \nabla g_m(x)
\]

This last equation is really \( n \) equations:
\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= \lambda_1 \frac{\partial g_1}{\partial x_1}, \\
&
\vdots \\
\frac{\partial f}{\partial x_n} &= \lambda_m \frac{\partial g_m}{\partial x_n}
\end{align*}
\]

So altogether we have \( n \times m \) equations in \( n \times m \) variables. One expects the solution set to be a finite set of points, which will include our desired max/min.

But why should the method work?

**Idea**
Say \( n = 3, m = 2 \) and we are trying to
\[
\text{max/min } f(x, y, z)
\]
on \( C = \{ g_1 = 0 \} \cap \{ g_2 = 0 \} \).

If \( \hat{r}(t) \) parametrizes \( C \), then \( \hat{r}(t) \) is tangent to \( C \), hence tangent to \( g_1 = 0 \) and \( g_2 = 0 \).

Thus \( \hat{r}' \perp \nabla g_1, \ nabla g_2 \).

At an extremum of \( f \) on \( C \) (or \( \hat{r}_0 = \hat{r}(t_0) \)),
\[
\mathbf{0} = \frac{d}{dt} f(\hat{r}(t)) \bigg|_{t=t_0} = \nabla f(\hat{r}_0) \cdot \hat{r}'(t_0)
\]

\( \Rightarrow \hat{r}' \perp \nabla f \).

Therefore, if \( \nabla g_1(\hat{r}_0) \) and \( \nabla g_2(\hat{r}_0) \) span the
\[\nabla f(\hat{r}_0) = \lambda_1 \nabla g_1(\hat{r}_0) + \lambda_2 \nabla g_2(\hat{r}_0)\]
for some \( \lambda_1, \lambda_2 \in \mathbb{R} \).

The \( \nabla g_1, \nabla g_2 \) will span as long as they are independent, i.e. \( \nabla g_1 \times \nabla g_2 \neq \mathbf{0} \).
General ansatz for Lagrange multipliers:

To find all candidates for extrema of \( f(x_1, \ldots, x_n) \) subject to \( m \) constraints \( g_1(x) = 0, g_2(x) = 0, \ldots, g_m(x) = 0 \), we need to solve the system

\[
\begin{align*}
\frac{\partial f}{\partial x_i} &= \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_i} \\
\frac{\partial g_j}{\partial x_i} &= 0, \quad j = 1, \ldots, m
\end{align*}
\]

(\( \star \)) this last equation is really in equations

\[
\begin{align*}
\frac{\partial f}{\partial x_1} &= \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_1} \\
\frac{\partial f}{\partial x_n} &= \sum_{j=1}^{m} \lambda_j \frac{\partial g_j}{\partial x_n}
\end{align*}
\]

So we really need to throw into the critical set, along with solutions to (\( \star \)), all points of \( g_1 = \cdots = g_m = 0 \) where \( \frac{\partial g_1}{\partial x_1}, \ldots, \frac{\partial g_m}{\partial x_n} \) are dependent.

But why should the method work?

IDEA

Say \( n=3, \ m=2 \) and we are trying to

\[
\min \text{ or } \max \ f(x_1, x_2, x_3)
\]

on

\[
C = \{ g_1 = 0 \} \cap \{ g_2 = 0 \}.
\]

If \( \hat{f}(t) \) parametrizes \( C \), then \( \hat{f}'(t) \) is tangent to \( C \), hence tangent to \( g_1=0 \) \& \( g_2=0 \).

Thus \( \hat{f}' \perp \nabla g_1 \& \nabla g_2 \).

At an extremum of \( f \) on \( C \) (or \( \hat{f}'(t_0) = 0 \)),

\[
0 = \Delta_t f(\hat{f}(t)) = \frac{df(\hat{f}(t_0))}{dt} \cdot \hat{f}'(t_0)
\]

\( \Rightarrow \hat{f}' \perp \nabla f \).

Therefore, if \( \nabla g_1(\hat{f}(t_0)) \& \nabla g_2(\hat{f}(t_0)) \) span the

subspace \( \mathbb{R}^2 \) to \( \hat{f}'(t_0) \), we have

\[
\nabla f(\hat{f}(t_0)) = \lambda_1 \nabla g_1(\hat{f}(t_0)) + \lambda_2 \nabla g_2(\hat{f}(t_0))
\]

for some \( \lambda_1, \lambda_2 \in \mathbb{R} \).

The \( \nabla g_1, \nabla g_2 \) will span as long as they are independent, i.e. \( \nabla g_1 \times \nabla g_2 \neq 0 \).
Ex 3] Optimize \( f(x, y, z) = x + 2y + 3z \)
on the ellipse that is the intersection of
the cylinder \( x^2 + y^2 = 2 \)
and the plane \( y + z = 1 \)
Ex 3] Optimize $f(x,y,z) = x + 2y + 3z$

on the ellipse that is the intersection of

the cylinder $x^2 + y^2 = 2$

and the plane $y + z = 1$

$g_1(x,y,z) = x^2 + y^2 - 2$

$g_2(x,y,z) = y + z - 1$

$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$
Ex 3] Optimize \( f(x, y, z) = x + 2y + 3z \)
on the ellipse that is the intersection of the cylinder \( x^2 + y^2 = 2 \) and the plane \( y + z = 1 \)

\( g_1(x, y, z) = x^2 + y^2 - 2 \)
\( g_2(x, y, z) = y + z - 1 \)
\( \nabla f = \lambda \nabla g_1 + \lambda_2 \nabla g_2 \Rightarrow (1, 2, 3) = \lambda (2x, 2y, 0) + \lambda_2 (0, 1, 1) \)

\[ \begin{cases} 1 = 2\lambda x \\ 2 = 2\lambda y + \lambda_2 \\ 3 = \lambda_2 \end{cases} \]
Ex 3] Optimize \( f(x,y,z) = x + 2y + 3z \)
on the ellipse that is the intersection of
the cylinder \( x^2 + y^2 = 2 \)
and the plane \( y + z = 1 \)
\[
\begin{align*}
g_1(x,y,z) &= x^2 + y^2 - 2 \
g_2(x,y,z) &= y + z - 1
\end{align*}
\]
\[\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \implies (1, 2, 3) = \lambda (2x, 2y, 0) + \mu (0, 1, 1)
\]
\[
\begin{cases}
1 = 2\lambda x \\
2 = 2\lambda y + \mu z \\
3 = \mu z
\end{cases}
\]
\[
\left( \frac{1}{2\lambda x} \right)^2 + \left( \frac{1}{\mu z} \right)^2 = 2
\]
Ex 3

Optimize \( f(x,y,z) = x + 2y + 3z \)
on the ellipse that is the intersection of the cylinder \( x^2 + y^2 = 2 \)
and the plane \( y + z = 1 \)

\( g_1(x,y,z) = x^2 + y^2 - 2 \quad \rightarrow \quad x^2 + y^2 = 2 \)

\( g_2(x,y,z) = y + z - 1 \)

\( \nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \quad \rightarrow \quad (1, 2, 3) = \lambda (2x, 2y, 0) + \mu (0, 1, 1) \)

\( \begin{align*}
1 &= 2\lambda x \\
2 &= 2\lambda y + \lambda_2 \\
3 &= \lambda_2
\end{align*} \)

\( \begin{align*}
\left( \frac{1}{2\lambda_1} \right)^2 + \left( \frac{1}{2\lambda_1} \right)^2 &= 2 \\
\left( \frac{1}{2\lambda_1} \right)^2 &= 2 \\
\frac{1}{4\lambda_1^2} &= 2 \\
\lambda_1 &= \pm \frac{1}{2} 
\end{align*} \)
Ex 3 | Optimize \( f(x,y,z) = x + 2y + 3z \)
on the ellipse that is the intersection of the cylinder \( x^2 + y^2 = 2 \) and the plane \( y + z = 1 \)

\[ g_1(x,y,z) = x^2 + y^2 - 2 \quad \rightarrow \quad x^2 + y^2 = 2 \]
\[ g_2(x,y,z) = y + z - 1 \]
\[ \nabla f = \lambda \nabla g_1 + \lambda_2 \nabla g_2 \quad \rightarrow \quad (1, 2, 3) = \lambda_1 (2x, 2y, 0) + \lambda_2 (0, 1, 1) \]
\[ \Rightarrow \begin{cases} 
1 = 2\lambda_1 x \quad \Rightarrow \quad x = \frac{1}{2\lambda_1} \\
2 = 2\lambda_1 y + \lambda_2 \\
3 = \lambda_2 
\end{cases} \quad \rightarrow \quad y = \frac{1}{2\lambda_1} \\
\left( \frac{1}{2\lambda_1}\right)^2 + \left( \frac{1}{2\lambda_1}\right)^2 = 2 \\
2 \quad \frac{1}{4\lambda_1^2} = 2 \\
\lambda_1 = \pm \frac{1}{2} \\
2 \quad \frac{1}{4\lambda_1^2} = 2 \\
\lambda_1 = \pm \frac{1}{2} \\
\lambda_2 \\
2 \quad \frac{1}{4\lambda_1^2} = 2 \\
\lambda_1 = \pm \frac{1}{2} \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
\lambda_2 \\
Ex 3] Optimize $f(x,y,z) = x + 2y + 3z$

on the ellipse that is the intersection of
the cylinder $x^2 + y^2 = 2$
and the plane $y + z = 1$

$g_1(x,y,z) = x^2 + y^2 - 2 \rightarrow x^2 + y^2 = 2$
$g_2(x,y,z) = y + z - 1$
$
\nabla f = \lambda \nabla g_1 + \lambda_2 \nabla g_2 \rightarrow (1, 2, 3) = \lambda_1 (2x, 2y, 0) + \lambda_2 (0, 1, 1)

\Rightarrow \begin{cases} 
1 = 2\lambda_1 x 
\Rightarrow x = \frac{1}{2\lambda_1} \\
2 = 2\lambda_1 y + \lambda_2 
\Rightarrow y = \frac{1 - \lambda_2}{2\lambda_1} \\
3 = \lambda_2
\end{cases}

\Rightarrow \left(\frac{1}{2\lambda_1}\right)^2 + \left(\frac{1 - \lambda_2}{2\lambda_1}\right)^2 = 2

\Rightarrow 2 \left(\frac{1}{2\lambda_1}\right)^2 = 2
\Rightarrow \lambda_1 = \pm \frac{1}{2}

2 cases:

- $\lambda_1 = \frac{1}{2} \Rightarrow (x, y, z) = (1, 1, 1)$
- $\lambda_1 = -\frac{1}{2} \Rightarrow (x, y, z) = (-1, 1, 0)$
Ex 3] Optimize \( f(x,y,z) = x + 2y + 3z \)
on the ellipse that is the intersection of the cylinder \( x^2 + y^2 = 2 \) and the plane \( y + z = 1 \).

\[
\begin{align*}
g_1(x,y,z) &= x^2 + y^2 - 2 
\rightarrow x^2 + y^2 = 2 \\
g_2(x,y,z) &= y + z - 1 \\
\nabla f &= \lambda \nabla g_1 + \mu \nabla g_2 
\rightarrow (1,2,3) = \lambda (2x,2y,0) + \mu (0,1,1) \\
\rightarrow \begin{cases} 
1 = 2\lambda x \rightarrow x = \frac{1}{2\lambda} \\
2 = 2\lambda y + \mu \\
3 = \mu 
\end{cases} 
\rightarrow y = \frac{-1}{2\lambda} \\
\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-1}{2\lambda}\right)^2 = 2 \\
2 \cdot \frac{1}{4\lambda^2} = 2 \\
\lambda = \pm \frac{1}{2}
\end{align*}
\]

2 cases:

- \( \lambda = \frac{1}{2} \Rightarrow (x,y,z) = (1,1,2) \Rightarrow f = 5 \) \( \text{MAX} \)
- \( \lambda = -\frac{1}{2} \Rightarrow (x,y,z) = (-1,1,0) \Rightarrow f = 1 \) \( \text{MIN} \)
Ex 3) Optimize \( f(x, y, z) = x + 2y + 3z \) on the ellipse that is the intersection of the cylinder \( x^2 + y^2 = 2 \) and the plane \( y + z = 1 \).

\[
\begin{align*}
g_1(x, y, z) &= x^2 + y^2 - 2 \
g_2(x, y, z) &= y + z - 1
\end{align*}
\]
\[
\vec{\nabla} f = \lambda \vec{\nabla} g_1 + \gamma \vec{\nabla} g_2 \implies (1, 2, 3) = \lambda (2x, 2y, 0) + \gamma (0, 1, 1)
\]
\[
\begin{cases}
1 = 2\lambda x \\
2 = 2\lambda y + \gamma \\
3 = \gamma
\end{cases} \implies y = \frac{1}{2\lambda}, \quad \frac{(\frac{1}{2\lambda})^2 + (\frac{1}{2\lambda})^2}{2} = 2
\]

2 cases:
- \( \lambda_1 = \frac{1}{2} \) \( \implies (x, y, z) = (1, -1, 2) \) \( \implies f = 5 \) \( \text{MAX} \)
- \( \lambda_1 = -\frac{1}{2} \) \( \implies (x, y, z) = (-1, 1, 0) \) \( \implies f = 1 \) \( \text{MIN} \)

Ex 4) Prove that the arithmetic mean of \( n \) nonnegative numbers is always larger than their geometric mean.
**Ex 3**] Optimize $f(x,y,z) = x + 2y + 3z$ on the ellipse that is the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $y + z = 1$.

\[ g_1(x,y,z) = x^2 + y^2 - 2 \implies x^2 + y^2 = 2 \]

\[ g_2(x,y,z) = y + z - 1 \]

\[ \nabla f = \lambda \nabla g_1 + \lambda_2 \nabla g_2 \implies (1,2,3) = \lambda(2x,2y,0) + \lambda_2(0,1,1) \]

\[ \begin{cases} 1 = 2\lambda_1 x \\ 2 = 2\lambda_1 y + \lambda_2 \\ 3 = \lambda_2 \end{cases} \implies \begin{cases} \frac{1}{2\lambda_1} = x \\ y = \frac{1}{2\lambda_1} \\ \lambda_2 = 3 \end{cases} \]

\[ \frac{(\frac{1}{2\lambda_1})^2}{(\frac{1}{2\lambda_1})^2} = 2 \]

\[ 2\frac{1}{4\lambda_1^2} = 2 \]

\[ \lambda_1 = \pm \frac{1}{2} \]

2 cases:

- $\lambda_1 = \frac{1}{2} \implies (x,y,z) = (1,1,2) \implies f = 5$ MAX

- $\lambda_1 = -\frac{1}{2} \implies (x,y,z) = (-1,1,0) \implies f = 1$ MIN

**Ex 4**] Prove that the arithmetic mean of $n$ nonnegative numbers is always larger than their geometric mean.

To do so, we shall maximize $\sqrt[n]{x_1 \cdots x_n}$ subject to $x_1 + \cdots + x_n = c$ ($g(x) = x_1 + \cdots + x_n - c$).
Ex 3] Optimize $f(x, y, z) = x + 2y + 3z$ on the ellipse that is the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $y + z = 1$.

$$g_1(x, y, z) = x^2 + y^2 - 2 \implies x^2 + y^2 = 2$$
$$g_2(x, y, z) = y + z - 1$$
$$\nabla f = \lambda \nabla g_1 + \lambda_2 \nabla g_2 \implies (1, 2, 3) = \lambda (2x, 2y, 0) + \lambda_2 (0, 1, 1)$$
$$\implies \begin{cases} 1 = 2\lambda_1 \implies x = \frac{1}{2\lambda_1} \\ 2 = 2\lambda_1 \implies y = \frac{1}{2\lambda_1} \\ 3 = \lambda_2 \end{cases}$$
$$\implies (\frac{1}{2\lambda_1})^2 + (\frac{1}{2\lambda_1})^2 = 2$$

2 cases:

- $\lambda_1 = \frac{1}{2} \implies (x, y, z) = (1, 1, 2) \implies f = 5 \text{ MAX}$
- $\lambda_1 = -\frac{1}{2} \implies (x, y, z) = (-1, 1, 0) \implies f = 1 \text{ MIN}$

Ex 4] Prove that the arithmetic mean of $n$ nonnegative numbers is always larger than their geometric mean.

To do so, we shall maximize $\sqrt[n]{x_1 \cdots x_n}$ subject to $x_1 + \cdots + x_n = c$ ($g(x) = x_1 + \cdots + x_n - c$).

Use $f(x) = x_1 \cdots x_n$.

$$\nabla f = \lambda \nabla g \implies (x_2 - x_n, x_3 - x_n, \ldots, x_{n-1} - x_n) = \lambda (1, 1, \ldots, 1).$$
Ex 3] Optimize \( f(x, y, z) = x + 2y + 3z \)
on the ellipse that is the intersection ofthe cylinder \( x^2 + y^2 = 2 \)and the plane \( y + z = 1 \)

\[ g_1(x, y, z) = x^2 + y^2 - 2 \quad \rightarrow \quad x^2 + y^2 = 2 \]
\[ g_2(x, y, z) = y + z - 1 \]

\[ \nabla f = \lambda \nabla g_1 + \lambda_2 \nabla g_2 \Rightarrow (1, 2, 3) = \lambda (2x, 2y, 0) + \lambda_2 (0, 1, 1) \]
\[ \begin{cases} 1 = 2\lambda x \Rightarrow x = \frac{1}{2\lambda} \\ 2 = 2\lambda y + \lambda_2 \\ 3 = \lambda_2 \end{cases} \quad \Rightarrow \quad y = \frac{1}{2\lambda} \]
\[ \left( \frac{1}{2\lambda} \right)^2 + \left( \frac{1}{2\lambda} \right)^2 = 2 \]
\[ 2 \left( \frac{1}{2\lambda} \right)^2 = 2 \quad \Rightarrow \quad \frac{1}{\lambda^2} = 2 \quad \lambda = \pm \frac{1}{2} \]

2 cases:

- \( \lambda_1 = \frac{1}{2} \Rightarrow (x, y, z) = (1, 1, 2) \Rightarrow f = 5 \) \( \text{MAX} \)
- \( \lambda_2 = -\frac{1}{2} \Rightarrow (x, y, z) = (-1, 1, 0) \Rightarrow f = 1 \) \( \text{MIN} \)

Ex 4] Prove that the arithmetic mean of \( n \) nonnegative numbers is always larger than their geometric mean.

To do so, we shall maximize \( \sqrt[n]{x_1 \cdots x_n} \) subject to \( x_1 + \cdots + x_n = c \) \( (g(x) = x_1 + \cdots + x_n - c) \).

Use \( f(x) = x_1 \cdots x_n \).

\[ \nabla f = \lambda \nabla g \Rightarrow (x_2 \cdots x_n, x_3 \cdots x_n, \ldots, x_{n-1} x_n) = \lambda (1, 1, \ldots, 1). \]

\[ x_2 \cdots x_n = \lambda \Rightarrow x_1 \cdots x_n = \lambda x_1 \]
\[ x_i x_2 \cdots x_n = \lambda \Rightarrow x_1 \cdots x_{i-1} x_{i+1} \cdots x_n = \lambda x_i \]
\[ \vdots \]
\[ x_1 \cdots x_{n-1} = \lambda \Rightarrow x_1 \cdots x_n = \lambda x_n \]
Ex 3) Optimize \( f(x, y, z) = x + 2y + 3z \)
on the ellipse that is the intersection of the cylinder \( x^2 + y^2 = 2 \) and the plane \( y + z = 1 \)

\[
\begin{align*}
g_1(x, y, z) &= x^2 + y^2 - 2 
\Rightarrow x^2 + y^2 = 2 \\
g_2(x, y, z) &= y + z - 1
\end{align*}
\]

\( \nabla f = \lambda \nabla g_1 + \lambda_2 \nabla g_2 \Rightarrow (1, 2, 3) = \lambda (2x, 2y, 0) + \lambda_2 (0, 1, 1) \)

\[
\begin{align*}
1 &= 2\lambda_1 x 
\Rightarrow x = \frac{1}{2\lambda_1} \\
2 &= 2\lambda_1 y + \lambda_2 
\Rightarrow y = \frac{1}{2\lambda_1} \\
3 &= \lambda_2 
\end{align*}
\]

\[
\left( \frac{1}{2\lambda_1} \right)^2 + \left( \frac{1}{2\lambda_1} \right)^2 = 2
\]

2 cases:

- \( \lambda_1 = \frac{1}{2} \Rightarrow (x, y, z) = (1, -1, 2) \Rightarrow f = 5 \) \( \text{MAX} \)
- \( \lambda_1 = -\frac{1}{2} \Rightarrow (x, y, z) = (-1, 1, 0) \Rightarrow f = 1 \) \( \text{MIN} \)

Ex 4) Prove that the arithmetic mean of \( n \) non-negative numbers is always larger than their geometric mean.

To do so, we shall maximize \( \sqrt[n]{x_1 \cdots x_n} \)
subject to \( x_1 + \cdots + x_n = c \) \( \left( g(x) = x_1 + \cdots + x_n - c \right) \).

Use \( f(x) = x_1 \cdots x_n \).

\[
\nabla f = \lambda \nabla g \Rightarrow (x_2 - x_n, x_3 - x_n, \ldots, x_{n-1} - x_n, 0) = \lambda (1, 1, \ldots, 1).
\]

\[
x_1 \cdots x_n = \lambda \Rightarrow x_1 \cdots x_n = \lambda x_1,
\]

\[
x_1 x_2 \cdots x_n = \lambda \Rightarrow x_2 \cdots x_n = \lambda x_2,
\]

\[
x_1 \cdots x_n = \lambda \Rightarrow x_1 \cdots x_n = \lambda x_n \]

Con't have \( \lambda = 0 \) if maximizing the product
Ex 3) Optimize $f(x,y,z) = x + 2y + 3z$ on the ellipse that is the intersection of the cylinder $x^2 + y^2 = 2$ and the plane $y + z = 1$.

\[ g_1(x,y,z) = x^2 + y^2 - 2 \Rightarrow x^2 + y^2 = 2 \]
\[ g_2(x,y,z) = y + z - 1 \]

\[ \nabla f = (\lambda, \lambda, \lambda) \quad \Rightarrow \quad (1, 2, 3) = (\lambda, \lambda, 0) + (0, 1, -1) \]
\[ \Rightarrow \begin{cases} 1 = 2\lambda \Rightarrow x = \frac{1}{2\lambda} \\ 2 = 2\lambda + \lambda \Rightarrow y = \frac{1}{2\lambda} \\ 3 = \lambda \Rightarrow z = \frac{3}{\lambda} \end{cases} \]

\[ \frac{1}{2\lambda} + \frac{1}{2\lambda} = 2 \]
\[ 2 \cdot \frac{1}{4\lambda} = 2 \quad \Rightarrow \lambda = \pm \frac{1}{2} \]

2 cases:
- $\lambda = \frac{1}{2} \Rightarrow (x, y, z) = (1, 1, 2) \Rightarrow f = 5$ MAX
- $\lambda = -\frac{1}{2} \Rightarrow (x, y, z) = (-1, 1, 0) \Rightarrow f = 1$ MIN

Ex 4) Prove that the arithmetic mean of $n$ nonnegative numbers is always larger than their geometric mean.

To do so, we shall maximize $\sqrt[n]{x_1 \cdots x_n}$ subject to $x_1 + \cdots + x_n = c$ ($g(x) = x_1 + \cdots + x_n - c$).

Use $f(x) = x_1 \cdots x_n$.

\[ \nabla f = \lambda \nabla g \Rightarrow (x_1, \ldots, x_n) = x_1, \ldots, x_n = (1, 1, \ldots, 1) \]

\[ x_1 \cdots x_n = \frac{c}{n} \quad \Rightarrow \quad x_1, \ldots, x_n = x_1, \ldots, x_n = \frac{c}{n} \]
\[ x_1 \cdots x_n = \frac{c}{n} \quad \Rightarrow \quad \lambda x_1, \ldots, \lambda x_n = \frac{c}{n} \]

So $x_1 = \cdots = x_n$. Since $x_1 + \cdots + x_n = c$, we get $x_j = \frac{c}{n}$ for each $j$. 
Ex 3] Optimize \( f(x,y,z) = x + 2y + 3z \) on the ellipse that is the intersection of the cylinder \( x^2 + y^2 = 2 \) and the plane \( y + z = 1 \).

\[ g_1(x,y,z) = x^2 + y^2 - 2 \quad \rightarrow \quad x^2 + y^2 = 2 \]
\[ g_2(x,y,z) = y + z - 1 \]

\[ \nabla f = \lambda \nabla g_1 + \gamma_2 \nabla g_2 \quad \rightarrow \quad (1,2,3) = \lambda (2z,2y,0) + \gamma_2 (0,1,1) \]

\[ \begin{cases} 1 = 2\lambda_1 x \\ 2 = 2\lambda_1 y + \lambda_2 \\ 3 = \lambda_2 \end{cases} \quad \rightarrow \quad y = \frac{2x}{\lambda_1}, \quad x = \frac{2}{\lambda_1} \]

\[ \left( \frac{1}{\lambda_1^2} \right)^2 + \left( \frac{1}{\lambda_1} \right)^2 = 2 \]

2 cases:

- \( \lambda_1 = \frac{1}{2} \Rightarrow (x,y,z) = (1,1,0) \Rightarrow f = 1 \quad \text{MIN} \)
- \( \lambda_1 = \frac{1}{2} \Rightarrow (x,y,z) = (-1,1,0) \Rightarrow f = 5 \quad \text{MAX} \)

Ex 4] Prove that the arithmetic mean of \( n \) nonnegative numbers is always larger than their geometric mean.

To do so, we shall maximize \( \sqrt[n]{x_1 \cdots x_n} \) subject to \( x_1 + \cdots + x_n = c \) (\( g(x) = x_1 + \cdots + x_n - c \)).

Use \( f(x) = x_1 \cdots x_n \).

\[ \nabla f = \lambda \nabla g \Rightarrow (x_2 - x_m, x_3 - x_m, \ldots, x_n - x_m) = \lambda (1,1,\ldots,1) \]

\[ x_2 \cdots x_n = \lambda \Rightarrow x_1 \cdots x_n = \lambda x_1 \]
\[ x_2 \cdots x_n = \lambda \Rightarrow x_2 \cdots x_n = \lambda x_2 \]
\[ \vdots \]
\[ x_1 \cdots x_n = \lambda \Rightarrow x_1 \cdots x_n = \lambda x_n \]

\[ \Rightarrow \lambda x_1 = \cdots = \lambda x_n \]

So \( x_1 = \cdots = x_n \). Since \( x_1 + \cdots + x_n = c \), we get \( x_j = \frac{c}{n} \) for each \( j \).

The maximum value of \( x_1 \cdots x_n \) is \( \frac{c^n}{n^n} \).

\[ \Rightarrow \text{maximum value of } \sqrt[n]{x_1 \cdots x_n} \text{ is } \frac{c}{n} \]

\[ \Rightarrow \sqrt[n]{x_1 \cdots x_n} \leq \frac{c}{n} = \frac{x_1 + \cdots + x_n}{n} \].
We finish by proving Theorem 1(i) from Lecture 32 in the special case where

\[ S = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]. \]

(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \), but we won't prove that.)
We finish by proving Theorem 1(i) from Lecture 32 in the special case where

\[ S = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]

(The following 3 results are still true on an arbitrary closed and bounded subset of \( \mathbb{R}^n \), but we won't prove that.)

For the following, let \( f : S \to \mathbb{R} \) be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max and a min

**Theorem C:** \( f \) is uniformly continuous
We finish by proving Theorem 1(i) from Lecture 32 in the special case where

\[ S = \prod_{i=1}^{n} [a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]. \]

(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \), but we won't prove that.)

For the following, let \( f : S \rightarrow \mathbb{R} \) be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

Chop \( S \) into half in each direction

\[ S = \bigcup_{a \in A} S_a \]

\[ |a_i| = 2^n \]

Side length: \( \frac{b_i - a_i}{2} \)
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = \prod[a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \).
(The following 3 results are still true on an arbitrary closed and bounded subset of \( \mathbb{R}^n \),
but we won't prove that.)

For the following, let \( f : S \rightarrow \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max and min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

Chop \( S \) into half in each direction.

\[ S = \bigcup_{a_i} S^{(i)} \]

And then into fours.

\[ \left| S^{(2)} \right| = 4^n \]

Side lengths \( \frac{b_i - a_i}{4} \).
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won’t prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

Chop \( S \) into half in each direction

\[ S = U_{d=0}^{\infty} S_d^{(2)} \]

and then into fourths

\[ = U_{d=0}^{\infty} S_d^{(2)} \]

and so on

\[ \left| a_m \right| = (2^m)^n \]

side lengths \( \frac{b_d - a_d}{2^m} \)
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b] \times [c, d] \times [e, f] \times \cdots \times [a_n, b_n] . \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \), but we won't prove that.)

For the following, let \( f: S \to \mathbb{R} \) be a continuous function.

**Theorem A**: \( f \) is bounded

**Theorem B**: \( f \) attains a max & a min

**Theorem C**: \( f \) is uniformly continuous

---

**Proof of A**: Assume otherwise.

Chop \( S \) into half in each direction:

\[ S = \bigcup_{a \in A} S_a \]

and then into fourths:

\[ S = \bigcup_{a \in A_2} S_{a_2} \]

and so on.

Then must be an \( a_m \in A_m \) for each \( m \) s.t. \( f|_{S_{a_m}} \) is unbounded, and we can choose these so that

\[ S \supset S_{a_1} \supset S_{a_2} \supset \cdots \supset S_{a_m} \supset \cdots \]

\[ (a_1, b_1) \supset (a_2, b_2) \supset \cdots \supset (a_m, b_m) \supset \cdots \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where

\[ S = [a, b] := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]

(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \), but we won't prove that.)

For the following, let \( f : S \rightarrow \mathbb{R} \) be a continuous function.

**Theorem A**: \( f \) is bounded

**Theorem B**: \( f \) attains a max & a min

**Theorem C**: \( f \) is uniformly continuous

---

**Proof of A**:

Assume otherwise. Chop \( S \) into half in each direction

\[ S = U \bigcup_{d \in \mathcal{D}} S_d^{(1)} \]

and then into fourths,

\[ S = U \bigcup_{d \in \mathcal{D}} S_d^{(2)} \]

and so on.

Then must be an \( d_m \in \mathcal{D}_m \) for each \( m \) s.t. \( f | S_m^{(m)} \) is unbounded, and we can choose these so that

\[ S_m^{(m)} \supset S_m^{(m-1)} \supset \cdots \supset S_m^{(1)} \supset S_m \]

So \( a_m \) resp. \( b_m \) are increasing resp. decreasing sequences with \( a_m \leq b_m \) \( \forall m \). (hence \( a_m \) resp. \( b_m \) is bounded above resp. below.)
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b]^n := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won’t prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

For each \( m \geq 1 \), \( S = \bigcup_{x \in \mathbb{R}^n} S_{x,m} \) and
then must be an \( d_m \in \mathbb{R}^n \)
for each \( m \) s.t. \( f|_{S_{d,m}} \) is
unbounded, and we can choose these so that
\[ d_1 > d_2 > d_3 > \cdots > d_m > \cdots. \]

So \( a_{d,m} \) resp. \( b_{d,m} \) are increasing resp. decreasing
sequences with \( a_{d,m} \leq b_{d,m} \), \( \forall m, k \)
(hence \( a_{d,m} \) resp. \( b_{d,m} \) is bounded above resp. below).
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won’t prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

For each \( m \geq 1 \), let \( S = \bigcup_{k=1}^{\infty} S^{(m)} \)
and for each \( m \), there must be an \( d_m \in S^{(m)}
for each \( m \).
If \( S^{(m)} \) is unbounded, and we can choose these so that
\[ S > S^{(1)} > S^{(2)} > \ldots \]

\[ [a_1, b_1] > [a_2, b_2] > \ldots \]

So \( a^{(m)}_{k} \), resp. \( b^{(m)}_{k} \) are increasing resp. decreasing
sequences, with \( a^{(m)}_{k} \leq b^{(m)}_{k} \) \( \forall m, k \)
(hence \( a^{(m)}_{k} \), resp. \( b^{(m)}_{k} \)
is bounded above resp. below). Since
\[ 0 \leq b^{(m)}_{k} - a^{(m)}_{k} \leq \frac{b_k - a_k}{2^m} \to 0 \text{ as } m \to \infty \]
we have \( \lim_{m \to \infty} a^{(m)}_{k} = \lim_{m \to \infty} b^{(m)}_{k} =: t_k \).
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = \prod_{i=1}^{n} [a_i, b_i] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won't prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

For each \( m \geq 1 \),
\[ S = \bigcup_{m=1}^{\infty} S_{a_m}^{(m)} \]
and then must be an \( a_m \in A_m \)
for each \( m \) s.t. \( f |_{S_{a_m}^{(m)}} \)
is unbounded, and we can choose these so that
\[ S \supset S_{a_1}^{(1)} \supset S_{a_2}^{(2)} \supset \cdots \supset S_{a_m}^{(m)} \supset \cdots \]

So \( a_m^{(m)} \) resp. \( b_m^{(m)} \) are increasing resp. decreasing
sequences with \( a_m^{(m)} \leq b_m^{(m)} \) \( \forall m \) (hence \( a_m^{(m)} \) resp.
\( b_m^{(m)} \) is bounded above resp. below). Since
\[ 0 \leq b_m^{(m)} - a_m^{(m)} \leq \frac{b_k - a_k}{2^m} = 0 \]
we have \( \lim_{m \to \infty} a_m^{(m)} = \lim_{m \to \infty} b_m^{(m)} =: t_k \).

By continuity of \( f \) at \( t \), \( \exists \varepsilon \geq 0 \) s.t.
\[ |f(x) - f(t)| < \varepsilon \quad \forall x \in B(t; \varepsilon) \cap S. \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b]^n = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won’t prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

For each \( m \geq 1 \), \( S = \bigcup_{a_m \in A_m} S_{a_m} \) and
there must be an \( a_m \in A_m \)
for each \( m \) s.t. \( f|_{S_{a_m}} \) is
unbounded, and we can choose these so that
\[ S = S_{a_1} \cup S_{a_2} \cup \cdots \cup S_{a_m} \cup \cdots. \]

So \( a_m \) resp. \( b_m \) are increasing resp. decreasing sequences with \( a_m \leq b_m \) \( \forall m \), (hence \( a_m \) resp.
\( b_m \) is bounded above resp. below). Since
\[ 0 \leq b_m - a_m \leq \frac{b_m - a_m}{2^m} \to 0 \]
we have \( \lim_{m \to \infty} a_m = \text{sup}_m a_m =: t \).

By continuity of \( f \) at \( t \), \( \exists \varepsilon > 0 \) s.t.
\[ |f(x) - f(t)| < 1 \quad \forall x \in B(t, \varepsilon) \cap S \]
\[ \Rightarrow |f(x)| < 1 + |f(t)| \quad \forall x \in B(t, \varepsilon) \cap S. \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b] = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \), but we won't prove that.)

For the following, let \( f : S \to \mathbb{R} \) be a continuous function.

Theorem A: \( f \) is bounded.

Theorem B: \( f \) attains a max & a min.

Theorem C: \( f \) is uniformly continuous.

Proof of A: Assume otherwise.

For each \( m \geq 1 \), \( S = \bigcup_{n \geq a_m} S(a_m) \), and there must be an \( a_m \in A_m \) for each \( n \) in \( \mathbb{N} \) if \( S(a_m) \) is unbounded, and we can choose these so that
\[ S > S(a_1) > S(a_2) > \cdots > S(a_m) > \cdots \]

So \( a_m \) resp. \( b_m \) are increasing resp. decreasing sequences with \( a_m \leq b_m < \frac{1}{2^m} \) \( \forall m \), hence \( a_m \) resp. \( b_m \) is bounded above resp. below. Since
\[ 0 \leq b_m - a_m \leq \frac{b_m - a_n}{2^m} \to 0 \quad m \to \infty \]
we have \( \lim_{m \to \infty} a_m = \lim_{m \to \infty} b_m = : t \).

By continuity of \( f \) at \( t \), \( \exists \epsilon > 0 \) s.t.
\[ |f(x) - f(t)| < \epsilon \quad \forall x \in B(t; \epsilon) \land S \]
\[ \Rightarrow |f(x)| < 1 + |f(t)| \]
\[ \Rightarrow f \text{ bounded on } B(t; \epsilon) \left( \Rightarrow S(a_m) \text{ for } m \geq 0 \right) \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b] := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(These following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \), but we won’t prove that.)

For the following, let \( f : S \rightarrow \mathbb{R} \) be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of A:** Assume otherwise.

For each \( m \geq 1 \), \( S = U \subset \mathcal{A}_m \mathcal{S}(m) \) and then must be an \( a_m \in \mathcal{A}_m \) for each \( m \) in such \( f \) is bounded, and we can choose these so that
\[ S > S^{(m)} = \begin{cases} \mathcal{A}_1 \mathcal{S}(m) \\ \mathcal{A}_2 \mathcal{S}(m) \\ \vdots \\ \mathcal{A}_m \mathcal{S}(m) \end{cases} \]

So \( a_{(m)} = b_{(m)} \) are increasing resp. decreasing sequences with \( a_{(m)} \leq b_{(m)} \) for all \( m \) (hence \( a_{(m)} \) resp. \( b_{(m)} \) is bounded above resp. below). Since
\[ 0 \leq b_{(m)} - a_{(m)} \leq \frac{b_k - a_k}{2^m} \rightarrow 0 \]

we have \( \lim_{m \rightarrow 0} a_{(m)} = \lim_{m \rightarrow 0} b_{(m)} = c = \epsilon_k \).

By continuity of \( f \) at \( \epsilon_k \), \( \exists \epsilon > 0 \) s.t.:
\[ |f(x) - f(\epsilon_k)| < 1 \quad \forall x \in B(\epsilon_k; \epsilon) \cap S \]

\[ \Rightarrow |f(x)| < 1 + |f(\epsilon_k)| \]

\[ \Rightarrow f \text{ bounded on } B(\epsilon_k; \epsilon) \Rightarrow S_{(m)}^{(m)} \text{ for } m \gg 0 \]

\[ \Rightarrow f \text{ bounded on } S_{(m)}^{(m)} \text{ for } m \gg 0 \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ \mathcal{S} = [a, b] := [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won’t prove that.)

For the following, let \( f: \mathcal{S} \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b] := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] . \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \), but we won't prove that.)

For the following, let \( f : S \rightarrow \mathbb{R} \) be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

Proof of B: Let \( M = \sup \{ f(x) | x \in S \} \) (= least upper bound for \( f \))

\[ g(\xi) := M - f(\xi) \geq 0 \text{ on } S . \]

Assume \( g(\xi) > 0 \) on \( S \).
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b]^n := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won’t prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of B:** let \( M = \sup \{ f(x) \mid x \in S \} \)
(= least upper bound for \( f \))

\[ g(\xi) := M - f(\xi) \geq 0 \text{ on } S. \]
Assume \( g(\xi) > 0 \text{ on } S. \)

Then \( \frac{1}{g} \) is continuous on \( S \)

\[ \Rightarrow \exists C \text{ s.t. } \frac{1}{g(\xi)} \leq C \forall \xi \in S. \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a,b] \times [c,d] \times \cdots \times [u,v] . \]
(The following 3 results are still true on an arbitrary closed and bounded subset of \( \mathbb{R}^n \), but we won't prove that.)

For the following, let \( f : S \to \mathbb{R} \) be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max and a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of B:** Let \( M = \sup \{ f(x) \mid x \in S \} \)

\[ (= \text{least upper bound for } f) \]

\[ g(x) := M - f(x) \geq 0 \text{ on } S. \]

Assume \( g(x) > 0 \text{ on } S. \)

Then \( \frac{1}{g} \) is continuous on \( S \)

\[ \Rightarrow \exists C \text{ s.t. } \frac{1}{g(x)} \leq C \forall x \in S \]

\[ \Rightarrow M - f(x) (\geq g) \geq \frac{1}{C} \quad \Rightarrow \]

\[ f(x) \leq M - \frac{1}{C} \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = \prod_{i=1}^{n} [a_i, b_i] \times [a_i, b_i] \times \ldots \times [a_i, b_i]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won’t prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A**: \( f \) is bounded

**Theorem B**: \( f \) attains a max & a min

**Theorem C**: \( f \) is uniformly continuous

---

**Proof of B**: let \( M = \sup \{ f(x) | x \in S \} \)
\( (= \text{least upper bound for } f) \)
\[ g(x) = M - f(x) \geq 0 \quad \text{on } S. \]

Assume \( g(x) > 0 \) on \( S \).

Then \( \frac{1}{g(x)} \) is continuous on \( S \)
\[ \Rightarrow \exists C \text{ s.t. } \frac{1}{g(x)} \leq C \quad \forall x \in S \]

**Theorem A**: \( g(x) \leq C \quad \forall x \in S \)
\[ \Rightarrow M - f(x) = g(x) \geq \frac{1}{C} \]
\[ \Rightarrow f(x) \leq M - \frac{1}{C} \]

Wrong
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\( S = [a, b] := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \).
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won't prove that.)

For the following, let \( f : S \to \mathbb{R} \)
be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

---

**Proof of B:** let \( M = \sup \{ f(x) \mid x \in S \} \)

\( (= \) least upper bound for \( f \) \)

\( g(\bar{x}) := M - f(\bar{x}) \geq 0 \) on \( S \).

**Assume** \( g(\bar{x}) > 0 \) on \( S \). **WRONG**

Then \( \frac{1}{g} \) is continuous on \( S \)

\[ \implies \exists C \text{ s.t. } \frac{1}{g(\bar{x})} \leq C \ \forall \bar{x} \in S \]

**Thm. A**

\[ \implies M - f(\bar{x}) (= g(\bar{x})) \geq \frac{1}{C} \]

\[ \implies f(\bar{x}) \leq M - \frac{1}{C} \]

**So** \( \exists \bar{x}_0 \in S \) s.t. \( g(\bar{x}_0) = 0 \)

\[ \implies f(\bar{x}_0) = M \geq f(\bar{x}) \ \forall \bar{x} \in S \]

\[ \implies f \text{ attains maximum at } \bar{x}_0. \]
We finish by proving Theorem 1(i) from Lecture 32 in the special case where
\[ S = [a, b]^n := [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]. \]
(The following 3 results are still true on an arbitrary closed & bounded subset of \( \mathbb{R}^n \),
but we won't prove that.)

For the following, let \( f : S \to \mathbb{R} \) be a continuous function.

**Theorem A:** \( f \) is bounded

**Theorem B:** \( f \) attains a max & a min

**Theorem C:** \( f \) is uniformly continuous

Proof of B: let \( M = \sup \{ f(x) \mid x \in S \} \)
\((= \text{least upper bound for } f)\)
\[ g(x) := M - f(x) \geq 0 \text{ on } S. \]

**Assume** \( g(x) > 0 \text{ on } S. \)

Then \( \frac{1}{g} \) is continuous on \( S \)

\[ \Rightarrow \exists C \text{ s.t. } \frac{1}{g(x)} \leq C \forall x \in S \]

\[ \Rightarrow M - f(x) (= g) \geq \frac{1}{C} \]

\[ \Rightarrow f(x) \leq M - \frac{1}{C} \]

So \( \exists x_0 \in S \text{ s.t. } g(x_0) = 0 \)

\[ \Rightarrow f(x_0) = M \geq f(x) \forall x \in S \]

\[ \Rightarrow f \text{ attains maximum at } x_0. \]

For min, apply argument to \(-f\).