Lecture 34

Line Integrals
A line integral is an integral over an oriented curve, which means a curve or "path" $C \subseteq \mathbb{R}^n$ with a choice of direction.
A line integral is an integral over an oriented curve, which means a curve $C \subseteq \mathbb{R}^n$ with a choice of direction.

We will need $C$ to be "piecewise smooth," for which my definition is a bit different from Apostol.

**Definition:** A **smooth parametrization** of $C$ is a 1-to-1, $C^1$ function $\vec{p}: [a,b] \rightarrow \mathbb{R}^n$ with image $C$, $\vec{p}(a) = A$ & $\vec{p}(b) = B$, and such that $\vec{p}'(t)$ is never 0 on $(a,b)$. $C$ is **smooth** if it has a smooth parametrization. (Apostol doesn't require)
A line integral is an integral over an oriented curve, which means a curve \( C \subseteq \mathbb{R}^n \) with a choice of direction.

Let \( f : \mathcal{D} \to \mathbb{R} \) be a function whose domain \( \mathcal{D} \subseteq \mathbb{R}^n \) contains \( C \).

We will need \( C \) to be "piecewise smooth," for which my definition is a bit different from Apostol.

Definition: A smooth parametrization of \( C \) is a 1-to-1, \( C^k \) function \( \hat{\gamma} : [a,b] \to \mathbb{R}^n \) with image \( \hat{C} \), \( \hat{\gamma}(a) = A \) & \( \hat{\gamma}(b) = B \), and such that \( \hat{\gamma}'(t) \) is never 0 on \( [a,b] \).

\( C \) is smooth if it has a smooth parametrization. Apostol doesn't require it.
A line integral is an integral over an oriented curve, which means a curve \( C \subseteq \mathbb{R}^n \) with a choice of direction.

We will need \( C \) to be “piecewise smooth,” for which my definition is a bit different from Apostol.

**Definition:** A smooth parametrization of \( C \) is a 1-to-1, \( C^1 \) function \( \mathbf{F} : [a,b] \rightarrow \mathbb{R}^n \) with image \( C \), \( \mathbf{F}(a) = A \) & \( \mathbf{F}(b) = B \), and such that \( \mathbf{F}'(t) \) is never 0 on \( [a,b] \).

\( C \) is smooth if it has a smooth parametrization. Apostol doesn’t require

Let \( f : \delta \rightarrow \mathbb{R} \) be a piecewise continuous function whose domain \( \delta \subseteq \mathbb{R}^n \) contains \( C \).

Think of \( f \) as having a graph “over” \( C \), and pretend we wanted to compute the area of the resulting “surface.”

**Intuition:** \( f \) is a density function (e.g., charge, mass, etc.) on a wire & we want the total (charge, mass, etc.)
A line integral is an integral over an oriented curve, which means a curve \( C \subseteq \mathbb{R}^n \) with a choice of direction.

We will need \( C \) to be “piecewise smooth,” for which my definition is a bit different from Apostol.

**Definition:** A smooth parametrization of \( C \) is a 1-to-1, \( C^1 \) function \( \vec{p} : [a,b] \to \mathbb{R}^n \) with image \( C \), \( \vec{p}(a) = A \) & \( \vec{p}(b) = B \), and such that \( \vec{p}'(t) \) is never 0 on \( [a,b] \). \( C \) is smooth if it has a smooth parametrization.

Let \( f : \mathcal{D} \to \mathbb{R} \) be a piecewise continuous function whose domain \( \mathcal{D} \subseteq \mathbb{R}^n \) contains \( C \):

Think of \( f \) as having a graph “over” \( C \), and pretend we wanted to compute the area of the resulting “surface.” Partitioning \( [a,b] \) by \( t_i = a + \frac{b-a}{n} i \) and \( C \) by \( \vec{p}_i(t) = \vec{p}(t_i) \), we have

\[
A \approx \sum_{i=1}^{\infty} f(\vec{p}(t_i)) \cdot \|\vec{p}'(t_i)\| \cdot \Delta t_i
\]
A line integral is an integral over an oriented curve, which means a curve \( C \subseteq \mathbb{R}^n \) with a choice of direction.

We will need \( C \) to be "piecewise smooth", for which my definition is a bit different from Apostol.

**Definition:** A smooth parametrization of \( C \) is a 1-to-1, \( C^1 \) function \( \tilde{p} : [a,b] \to \mathbb{R}^n \) with image \( C \), \( \tilde{p}(a) = A \) \& \( \tilde{p}(b) = B \), and such that \( \tilde{p}'(t) \) is never 0 on \([a,b] \). \( C \) is smooth if it has a smooth parametrization. Apostol doesn't require

Let \( f : S \to \mathbb{R} \) be a piecewise continuous function whose domain \( S \subseteq \mathbb{R}^n \) contains \( C \).

Think of \( f \) as having a graph "over" \( C \), and pretend we wanted to compute the area of the resulting "surface". Partitioning \([a,b]\) by \( t_i = a + \frac{b-a}{n} i \) and \( C \) by \( \tilde{x}_i = \tilde{p}(t_i) \), we have

\[
A = \sum_{i=1}^{n} f(\tilde{x}_i)(\Delta s_i) = \sum_{i=1}^{n} f(\tilde{x}(c_i))|\tilde{p}'(c_i)(\Delta c_i)|,
\]

which in the limit gives \( (A = \) \[
\int_C f \, ds := \int_a^b f(\tilde{x}(t)) |\tilde{p}'(t)| \, dt,
\]

the line integral of \( f \) with respect to arc length.
Ex 1: \( \int_C 1 \, ds = \int_a^b \| \dot{r}(t) \| \, dt \) is just the arclength! 

Let \( f: \mathcal{S} \to \mathbb{R} \) be a piecewise continuous function whose domain \( \mathcal{S} \subseteq \mathbb{R}^n \) contains \( C \):

Think of \( f \) as having a graph "over" \( C \), and pretend we wanted to compute the area of the resulting "surface". Partitioning \([a, b]\) by \( t_i = a + \frac{i(b-a)}{n} \) and \( C \) by \( x^{(i)} = \dot{r}(t_i) \), we have

\[
A = \sum_{i=1}^{n} f(x^{(i)}) \Delta s_i = \sum_{i=1}^{n} f(\dot{r}(t_i)) \| \dot{r}(t_i) \| dt_i(\Delta s_i),
\]

which in the limit gives (\( A = \) )

\[
\int_C f \, ds := \int_a^b f(\dot{r}(t)) \| \dot{r}(t) \| \, dt = \text{the line integral of } f \text{ with respect to arclength},
\]

\( \text{e.g., } "ds" \)
Ex 1/ \( \int_C 1 \, ds = \int_a^b \| \vec{r}'(t) \| \, dt \) is just the arc length!

Ex 2/ Evaluate \( \int_C x^2y \, ds \), where \( C \) is the quarter-circle.

\[ A = (3,0), B = (0,3) \]

Let \( f : \mathbb{S} \to \mathbb{R} \) be a piecewise continuous function whose domain \( \mathbb{S} \subseteq \mathbb{R}^n \) contains \( C \):

Think of \( f \) as having a graph “over” \( C \), and pretend we wanted to compute the area of the resulting “surface”. Partitioning \([a,b] \) by \( t_i = a + \frac{b-a}{n} i \) and \( C \) by \( \vec{x}^{(i)} = \vec{r}(t_i) \), we have

\[ A = \sum_{i=1}^{n} f(\vec{x}^{(i)}(\Delta s)) \approx \sum_{i=1}^{n} f(\vec{r}(t_i)) \| \vec{r}'(t_i) \| \Delta s \],

which in the limit gives (\( A = \))

\[ \int_C f \, ds := \int_a^b f(\vec{r}(t)) \| \vec{r}'(t) \| \, dt \text{, e.h.e. “ds”} \]

the line integral of \( f \) with respect to arc length.
Ex 1 / \( \int_C 1 \, ds = \int_a^b \| \vec{t} \| \, dt \) is just the arclength! //

Ex 2 / Evaluate \( \int_C x^2 y \, ds \), where \( C \) is the quarter-circle \( B = (0,3) \) \( A = (3,0) \)

- \( \vec{r}(t) = (3 \cos t, 3 \sin t) \) on \( t \in [0, \pi/2] \)
- \( \int_C x^2 y \, ds = \int_0^{\pi/2} 27 \cos^2 t \sin t \sqrt{(-3 \sin t)^2 + (3 \cos t)^2} \, dt 
  = 81 \int_0^{\pi/2} \cos^2 t \sin t \, dt = 81 \left[ -\frac{1}{3} \cos^3 t \right]_0^{\pi/2} 
  = 27 \)

Think of \( f \) as having a graph “over” \( C \), and pretend we wanted to compute the area of the result of the resulting “surface”. Partitioning \([a,b]\) by \( t_i : a + \frac{b-a}{n}, i \) and \( C \) by \( \vec{r}(t_i) \), we have

\[
A \approx \sum_{i=1}^n f(\vec{r}(t_i)) \Delta s_i \approx \sum_{i=1}^n f(\vec{r}(t_i)) \| \vec{r}'(t_i) \| \Delta t_i $

which in the limit gives \( (A = \int_C f(x) \, ds) \)

\[
\int_C f \, ds := \int_a^b f(\vec{r}(t)) \| \vec{r}'(t) \| \, dt, \quad \text{c.o.e. “ds”}
\]

the line integral of \( f \) with respect to arclength.
Ex 1 / $\int_C 1 \, ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$ is just the arclength.

Ex 2 / Evaluate $\int_C x^2 y \, ds$, where $C$ is the quarter-circle $A = (3,0)$, $B = (0,3)$.

- $\vec{r}(t) = (3 \cos t, 2 \sin t)$ on $t \in [0, \pi/2]$
  \[ \int_C x^2 y \, ds = \int_0^{\pi/2} 27 \cos^2 t \sin t \sqrt{(3 \sin t)^2 + (3 \cos t)^2} \, dt = 27 \int_0^{\pi/2} \cos^2 t \, dt = 27 \left[ \frac{1}{2} \cos^2 t \right]_0^{\pi/2} = \frac{27}{2} \]

- $\vec{r}(t) = (\sqrt{9 - t^2}, t)$ on $t \in [0, 3]$
  \[ \int_C x^2 y \, ds = \int_0^3 (9 - t^2) t \sqrt{\left(\frac{-2t}{\sqrt{9-t^2}}\right)^2 + 1^2} \, dt = \int_0^3 3t \sqrt{9 - t^2} \, dt \]
  \[ = \left[ -(9 - t^2)^{3/2} \right]_0 = 9^{3/2} = 27. \]

Let $f : \partial \Omega \to \mathbb{R}$ be a piecewise continuous function whose domain $\partial \Omega \subseteq \mathbb{R}^2$ contains $C$.

Think of $f$ as having a graph “over” $C$, and pretend we wanted to compute the area of the resulting “surface”. Partitioning $[a,b]$ by $t_i = a + \frac{b-a}{n}i$ and $C$ by $x^{(i)} = \vec{r}(t_i)$, we have

\[ A = \sum_{i=1}^n f(x^{(i)}) \Delta s_i = \sum_{i=1}^n f(\vec{r}(t_i)) \|\vec{r}'(t_i)\| \Delta t_i, \]

which in the limit gives \( A = \int_C f \, ds \) (c.f. “ds”)

\[ \int_C f \, ds := \int_a^b f(\vec{r}(t)) \|\vec{r}'(t)\| \, dt, \]

the line integral of $f$ with respect to arclength.
The fact that the two parametrizations in Ex. 2 yield the same answer is a consequence of the fact that (\(\star\)) can be described as a limit of Riemann sums with no reference to a specific parametrization.

\[
\int_C x^2 y \, ds := \int_0^1 f(r(t)) ||r'(t)|| \, dt \\
\text{Ex 2: Evaluate } \int_C x^2 y \, ds, \text{ where } C \text{ is the quarter-circle } B=(0,3), A=(3,0)
\]

- \( \vec{r}(t) = (3 \cos t, 2 \sin t) \) on \( t \in [0, \pi/2] \)
  \[
  \int_C x^2 y \, ds = \int_0^{\pi/2} 27 \cos^2 t \sin t \sqrt{(3 \sin t)^2 + (3 \cos t)^2} \, dt \\
  = 81 \int_0^{\pi/2} \cos^2 t \sin t \, dt = 81 \left[ -\frac{1}{3} \cos^3 t \right]_0^{\pi/2} \\
  = 27
  \]

- \( \vec{r}(t) = (\sqrt{9-t^2}, t) \) on \( t \in [0, 3] \)
  \[
  \int_C x^2 y \, ds = \int_0^3 (9-t^2) \sqrt{\left(\frac{-2t}{\sqrt{9-t^2}}\right)^2 + 1^2} \, dt \\
  = \int_0^3 3t \sqrt{9-t^2} \, dt \\
  = \left[ -(9-t^2)^{3/2} \right]_0^3 = 9^{3/2} = 27
  \]
Ex 2/ Evaluate $\int_C x^2 y \, ds$, where $C$ is the quarter-circle $B=(0,3)$,
$A=(3,0)$.

- $\hat{r}(t) = (3 \cos t, 2 \sin t)$ on $t \in [0, \frac{\pi}{2}]$

\[
\int_C x^2 y \, ds = \int_0^{\frac{\pi}{2}} 27 \cos^2 t \sin t \sqrt{(-3 \sin t)^2 + (2 \cos t)^2} \, dt
= 81 \int_0^{\frac{\pi}{2}} \cos^2 t \sin t \, dt = 81 \left[ \frac{1}{3} \cos^3 t \right]_0^{\frac{\pi}{2}}
= 27
\]

- $\hat{r}(t) = (\sqrt{9-t^2}, t)$ on $t \in [0, 3]$

\[
\int_C x^2 y \, ds = \int_0^3 (9-t^2) t \sqrt{\left( \frac{2t}{\sqrt{9-t^2}} \right)^2 + 1^2} \, dt
= \int_0^3 3t \sqrt{9-t^2} \, dt
= \left[ -(9-t^2)^{\frac{3}{2}} \right]_0^3 = 9^{\frac{3}{2}} = 27.
\]

The fact that the two parametrizations in Ex. 2 yield the same answer is a consequence of the fact that $(*)$ can be described as a limit of Riemann sums with no reference to a specific parametrization. $\hat{r} : [c, d] \to \mathbb{R^n}$

Alternatively, if $\hat{r}(t)$ and $\hat{r}(u)$ are two smooth parametrizations of $C$, we can write $\hat{r}(u) = \hat{r}(g(u))$ (with $g \in C^1$) and then

\[
\int_C f(\hat{r}(u)) \|\hat{r}'(u)\| \, du
= \int_a^b f(\hat{r}(g(u))) \|\hat{r}'(g(u))\| g'(u) \, du
= \int_a^b f(\hat{r}(t)) \|\hat{r}'(t)\| \, dt
\]

This shows that $(*)$ is independent of the choice of parametrization.
\[ \mathcal{C} f \, ds := \int_{a}^{b} f(\mathbf{r}(t)) \| \mathbf{r}'(t) \| \, dt \] 

**Ex 2**/ Evaluate \( \int_{\mathcal{C}} x^2y \, ds \), where \( \mathcal{C} \) is the quarter-circle \( B = (0,3) \) \( A = (3,0) \)

- \( \mathbf{r}(t) = (3 \cos t, 2 \sin t) \) on \( t \in [0, \pi/2] \)
  \[
  \int_{\mathcal{C}} x^2y \, ds = \int_{0}^{\pi/2} 27 \cos^2 t \sin t \sqrt{1-3 \sin^2 t \cos^2 t} \, dt \\
  = 81 \int_{0}^{\pi/2} \cos^2 t \sin t \, dt = 81 \left[ -\frac{1}{3} \cos^3 t \right]_{0}^{\pi/2} \\
  = 27
  
- \( \mathbf{r}(t) = (\sqrt{9-t^2}, t) \) on \( t \in [0,3] \)
  \[
  \int_{\mathcal{C}} x^2y \, ds = \int_{0}^{3} (9-t^2) t \sqrt{\left(\frac{t}{\sqrt{9-t^2}}\right)^2 + 1^2} \, dt \\
  = \int_{0}^{3} 3t \sqrt{9-t^2} \, dt \\
  = \left[ -\left(9-t^2\right)^{3/2} \right]_{0}^{3} = 9^{3/2} = 27.
  
The fact that the two parametrizations in Ex. 2 yield the same answer is a consequence of the fact that (1) can be described as a limit of Riemann sums with no reference to a specific parametrization.

Alternatively, if \( \mathbf{r}(t) \) and \( \mathbf{R}(u) \) are two smooth parametrizations of \( \mathcal{C} \), we can write \( \mathbf{R}(u) = \mathbf{r}(g(u)) \) (with \( g \in C^1 \)) and then

\[
\int_{\mathcal{C}} f(\mathbf{r}(u)) \| \mathbf{r}'(u) \| \, du \\
= \int_{\mathcal{C}} f(\mathbf{r}(u)) \| \mathbf{r}'(g(u)) \| g'(u) \, du \\
= \int_{a}^{b} f(\mathbf{r}(t)) \| \mathbf{r}'(t) \| \, dt = \int_{a}^{b} \ldots \\
= g'(u) \, du \text{ shows that (1) is independent of the choice of parametrization.}

In fact, the orientation doesn't even matter because if \( g \) sends \( t \rightarrow a + \int_{a}^{t} \ldots \) the sign changes cancel.
Ex 3/ Compute \( \int_C \mathbf{z} \cdot ds \), where \( C \) is the line segment in \( \mathbb{R}^3 \) from \((3,0,1)\) to \((5,1,7)\).

The fact that the two parametrizations in Ex. 2 yield the same answer is a consequence of the fact that (\#) can be described as a limit of Riemann sums with no reference to a specific parametrization. \( \mathbf{r} : [c,d] \rightarrow \mathbb{R}^3 \)

Alternatively, if \( \mathbf{r}(t) \) and \( \mathbf{r}(u) \) are two smooth parametrizations of \( C \), we can write \( \mathbf{r}(u) = \mathbf{i}(g(u)) \) (with \( g \in \mathcal{C}^1 \)) and then
\[
\int_C f(\mathbf{r}(u)) \, ds = \int_C f(\mathbf{r}(u)) \, ||\mathbf{r}'(u)|| \, du
\]
\[
= \int_C f(\mathbf{i}(g(u))) \, ||\mathbf{i}'(g(u))|| \, |g'(u)| \, du
\]
\[
= \int_C f(\mathbf{r}(t)) \, ||\mathbf{r}'(t)|| \, dt = \int_a^b ...
\]

Show that (\#) is independent of the choice of parametrization.

In fact, the orientation doesn’t even matter because if \( g \) sends \( \mathcal{C} \) to \( [c,d] \), the sign changes cancel.
The fact that the two parametrizations in Ex. 2 yield the same answer is a consequence of the fact that (*) can be described as a limit of Riemann sums with no reference to a specific parametrization. 

Alternatively, if \( \vec{r}(t) \) and \( \vec{r}'(u) \) are two smooth parametrizations of \( C \), we can write \( \vec{r}(u) = \vec{r}(g(u)) \) (with \( g \in C^1 \)) and then 

\[
\int_{C} f(\vec{r}(u)) \| \vec{r}'(u) \| \, du 
= \int_{a}^{b} f(\vec{r}(g(t))) \| \vec{r}'(g(t)) \| g'(t) \, dt 
= -\int_{a}^{b} f(\vec{r}(t)) \| \vec{r}'(t) \| \, dt = \int_{a}^{b} \ldots 
\]

Since \( g' \) is positive, \( \int_{a}^{b} g'(u) \, du \) shows that (*) is independent of the choice of parametrization.

In fact, the orientation doesn't even matter because if \( g \) sends \( \vec{C} \rightarrow [a, b] \), the sign changes cancel.
Ex 3/ Compute \( \int_C z \, ds \), where \( C \) is the line segment in \( \mathbb{R}^3 \) from \((3,0,1)\) to \((5,1,7)\).

Put \( \mathbf{r}(t) = (1-t)(3,0,1) + t(5,1,7), \quad t \in [0,1]. \)

Then \( \int_C z \, ds = \int_0^1 (1+6t) \sqrt{2^2 + 1^2 + 6^2} \, dt = \sqrt{41} [t+3t^2]_0^1 = 4\sqrt{41}. \)

\[ \mathbf{r}'(t) = (5,1,7) - (3,0,1) = (2,1,6) \]

The fact that the two parametrizations in Ex. 2 yield the same answer is a consequence of the fact that \((*)\) can be described as a limit of Riemann sums with no reference to a specific parametrization. \( \mathbf{r}: [c,d] \to \mathbb{R}^3 \)

Alternatively, if \( \mathbf{r}(t) \) and \( \mathbf{R}(u) \) are two smooth parametrizations of \( C \), we can write \( \mathbf{R}(u) = \mathbf{r}(g(u)) \) (with \( g \in C^1 \)) and then

\[ \int_C f(\mathbf{R}(u)) \, ||\mathbf{R}'(u)|| \, du \]

\[ = \int_c^d f(\mathbf{r}(g(u))) \, ||\mathbf{r}'(g(u))|| \, g'(u) \, du \]

\[ = -\int_c^d f(\mathbf{r}(t)) \, ||\mathbf{r}'(t)|| \, dt = \int_c^d -f(\mathbf{r}(t)) \, ||\mathbf{r}'(t)|| \, dt \]

\[ = \int_c^d -f(\mathbf{r}(t)) \, ||\mathbf{r}'(t)|| \, dt = \int_c^d \text{ shows that (*) is independent of the choice of parametrization.} \]

In fact, the orientation doesn't even matter because if \( g \) sends \([a,b] \to [c,d] \)

the sign changes cancel.
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \sqrt{\dot{\mathbf{r}}(t) \cdot \dot{\mathbf{r}}(t)} \, dt \]  

Ex 3/ Compute \( \int_C z \, ds \), where \( C \) is the line segment in \( \mathbb{R}^3 \) from \((3,0,1)\) to \((5,1,7)\).

Put \( \mathbf{r}(t) = (1-t)(3,0,1) + t(5,1,7), \quad t \in [0,1] \).

Then \( \int_C z \, ds = \int_0^1 1 + 6t \sqrt{2^2 + 1^2 + 6^2} \, dt = \frac{5}{4} \sqrt{41} \) \( \int_0^1 [t+3z^2] \)
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \, ||\mathbf{r}'(t)|| \, dt \] (x)

Ex 3/ Compute \( \int_C z \, ds \), where \( C \) is the line segment in \( \mathbb{R}^3 \) from \((3,0,1)\) to \((5,1,7)\).

Put \( \mathbf{r}(t) = (1-t)(3,0,1) + t(5,1,7) \), \( t \in [0,1] \).

Then \( \int_C z \, ds = \int_0^1 (1+6t) \sqrt{2^2+1^2+6^2} \, dt = \sqrt{41} \left[ t+3t^2 \right]_0^1 = 4\sqrt{41} \).

Besides the line integral of \( f(\mathbf{x}) \) with respect to arc length, we can integrate with respect to one of the coordinates \( x_i \):

\[ \int_C f \, dx_i := \int_0^b f(\mathbf{r}(t)) \, \mathbf{r}_i'(t) \, dt \]

\( x_i = \mathbf{r}_i(t) \) has the component \( x_i = r_i(t) \).

In 2 or 3 variables then one writes

\[ \int_C f \, dx, \int_C f \, dy, \int_C f \, dz \] .

Ex 4/ Evaluate the line integral

\[ \int_C (x^2-y^2) \, dx + \int_C 2xy \, dy \]

along the curve parametrized by \( \mathbf{r}(t) = (t^2, t^3) \), \( t \in [0,1] \).
In 2 or 3 variables then can written
\[ \int_C f \, dx, \int_C f \, dy, \int_C f \, dz. \]

Ex 4/ Evaluate the line integral
\[ \int_C (x^2 - y^2) \, dx + 2xy \, dy \]
along the curve parametrized by \( \vec{r}(t) = (t^2, t^3) \) for \( t \in [0, 1] \).

Besides the line integral of \( f(\vec{r}) \) with respect
to arc-length, we can integrate with respect to
one of the coordinates \( x_i \):

\[ \int_C f \, dx_i = \int_a^b f(\vec{r}(t)) \, r_i'(t) \, dt \]

where \( r_i = r_i(t) \) has its component \( x_i = r_i(t) \).
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt \] 

(1)

In 2 or 3 variables, then it is written
\[ \int_C f \, dx, \int_C f \, dy, \int_C f \, dz. \]

Ex 3/ Compute \( \int_C z \, ds \), where \( C \) is the line segment in \( \mathbb{R}^3 \) from \((3,0,1)\) to \((5,1,7)\).

Put \( \mathbf{r}(t) = (1-t)(3,0,1) + t(5,1,7), \ t \in [0,1]. \)

Then \( \int_C z \, ds = \int_0^1 (1+6t) \sqrt{2^2+1^2+6^2} \, dt = \sqrt{41} \int_0^1 (1+3t^2) \, dt = 4\sqrt{41}. \)

\[ z = (1-t)1 + t7 = 1+6t \]
\[ \mathbf{r}'(t) = (5,1,7) - (3,0,1) = (2,1,6) \]

Ex 4/ Evaluate the line integral
\[ \int_C (x^2 - y^2) \, dx + 2xy \, dy \]

along the curve parametrized by \( \mathbf{r}(t) = (t^2, t^3), \ t \in [0,1]. \)

\[ \begin{align*}
\int_C (x^2 - y^2) \, dx & = \int_0^1 (t^4 - t^6) \, 2t \, dt \\
\int_C 2xy \, dy & = \int_0^1 (t^5 - t^8) \, 3t^2 \, dt 
\end{align*} \]

Besides the line integral of \( f(\mathbf{r}) \) with respect to arc length, we can integrate with respect to one of the coordinates \( x; \)

\[ \int_C f \, dx := \int_a^b f(\mathbf{r}(t)) \, dx \]

\[ \text{where } \mathbf{r} = \mathbf{r}(t) \text{ has its component } x = r_1(t) \]
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \sqrt{\mathbf{r}'(t)^2} \, dt \]  

Ex 3/ Compute \( \int_C z \, ds \), where \( C \) is the line segment in \( \mathbb{R}^3 \) from \((3,0,1)\) to \((5,1,7)\).

Put \( \mathbf{r}(t) = (1-t)(3,0,1) + t(5,1,7) \), \( t \in [0,1] \).

Then \( \int_C z \, ds = \int_0^1 (1+6t) \sqrt{2^2 + 1^2 + 6^2} \, dt = \sqrt{41} \left[ t + 3t^2 \right]_0^1 = 4 \sqrt{41} \).

\[ \mathbf{z} = (1-t)\mathbf{1} + t\mathbf{7} = 1+6t \]
\[ \mathbf{r}'(t) = (5,1,7)-(3,0,1) = (2,1,6) \]

Besides the line integral of \( f(\mathbf{z}) \) with respect to arc length, we can integrate with respect to one of the coordinates \( x_i \):

\[ \int_C f \, dx_i := \int_a^b f(\mathbf{r}(t)) \mathbf{r}'(t) \, dt \]

\[ dx_i = \frac{\mathbf{z}}{\sqrt{\mathbf{z} \cdot \mathbf{z}}} \text{ has its component } x_i = r_i(t) \]

Ex 4/ Evaluate the line integral

\[ \int_C (x^2 - y^2) \, dx + 2xy \, dy \]

Combining them like this is typical notation along the curve parametrized by \( \mathbf{r}(t) = (t^2, t^3) \), \( t \in [0,1] \).

\[ dx = d(t^2) = 2t \, dt, \quad dy = d(t^3) = 3t^2 \, dt \]

\[ \Rightarrow \int_C \ldots = \int_0^1 \left( (t^4) - (t^3) \right) 2t \, dt + 7t^6 \, dt \]
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt \]  

(4)

Ex 3/ Compute \( \int_C z \, ds \), where \( C \) is the line segment in \( \mathbb{R}^3 \) from \((3,0,1)\) to \((5,1,7)\).

Put \( \mathbf{r}(t) = (1-t)(3,0,1)+t(5,1,7), \ t \in [0,1] \).

Then \( \int_C z \, ds = \int_0^1 (1+6t) \sqrt{2^2+1^2+6^2} \, dt = 4\sqrt{11} \)

\[ = 4\sqrt{41}. \]

\[ z = (1-t)\cdot 1 + t \cdot 7 = 1+6t \]

\[ \mathbf{r}'(t) = (5,1,7)-(3,0,1) = (2,1,6) \]

Besides the line integral of \( f(z) \) with respect to arc length, we can integrate with respect to one of the coordinate axes:

\[ \int_C f \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt \]

\[ \int_C f \, dx; \quad \int_C f \, dy, \quad \int_C f \, dz. \]

Ex 4/ Evaluate the line integral

\[ \int_C (x^2-y^2) \, dx + 2xy \, dy \]

Combining them like this is typical notation along the curve parameterized by \( \mathbf{F}(t) = (t^2, t^3) \), \( t \in [0,1] \).

\[ dx = d(t^2) = 2t \, dt, \quad dy = d(t^3) = 3t^2 \, dt \]

\[ \int_C \cdots = \int_0^1 \frac{(t^3)^2-(t^2)^2}{2t} \, dt = 2 \int_0^1 t^2 \, dt + 3 \int_0^1 t^5 \, dt \]

\[ = \int_0^1 (2t^5-2t^3+6t^3) \, dt = \left[ \frac{1}{3} t^6 + \frac{1}{2} t^8 \right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}. \]
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \left\| \mathbf{r}'(t) \right\| \, dt \quad (\star) \]

\[ \int_C f \, dx := \int_a^b f(\mathbf{r}(t)) \mathbf{r}'(t) \, dt \quad (***) \]

**Properties:**

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In 2 or 3 variables, this is written as:

\[ \int_C f \, dx, \int_C f \, dy, \int_C f \, dz. \]

**Ex 4/ Evaluate the line integral**

\[ \int_C (x^2 - y^2) \, dx + 2xy \, dy \]

Combining terms like this is typical notation along the curve parametrized by \( \mathbf{r}(t) = (t^3, t^2) \), \( t \in [0, 1] \).

\[ d\mathbf{r} = \mathbf{r}'(t) \, dt = \left( 3t^2, 2t \right) \, dt. \]

\[ d\mathbf{r} = 2t \, dt, \quad dy = dt \]

\[ \int_C (x^2 - y^2) \, dx + 2xy \, dy = \int_0^1 \left( (t^3)^2 - (t^2)^2 \right) 2t \, dt + 2t^2 \cdot t^3 \cdot 2t \, dt \]

\[ = \int_0^1 \left( 2t^7 - 2t^5 + 2t^3 \right) \, dt \]

\[ = \int_0^1 \left( \frac{1}{3} t^6 + \frac{1}{2} t^8 + \frac{2}{3} t^3 \right) \, dt \]

\[ = \left[ \frac{1}{3} \frac{1}{7} t^7 + \frac{1}{2} \frac{1}{9} t^9 + \frac{2}{3} \frac{1}{4} t^4 \right]_0^1 = \frac{1}{21} + \frac{1}{18} + \frac{1}{6} = \frac{5}{6}. \]
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \| \mathbf{r}'(t) \| \, dt \] (\star)

\[ \int_C f \, dx := \int_a^b f(\mathbf{r}(t)) \mathbf{r}'(t) \, dt \] (\star\star)

Properties:

1. \[ C = C_1 + C_2 \Rightarrow \int_C f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds \]


In 2 or 3 variables then can write

\[ \int_C f \, dx, \int_C f \, dy, \int_C f \, dz. \]

Ex 4/ Evaluate the line integral

\[ \int_C (x^2 - y^2) \, dx + 2xy \, dy \]

Combining them like this is typical notation

along the curve parameterized by \( \mathbf{r}(t) = (t^3, t^2) \), \( t \in [0, 1] \).

\[ dx = d(t^3) = 2t \, dt, \quad dy = d(t^2) = 3t \, dt \]

\[ \int_C (x^2 - y^2) \, dx + 2xy \, dy = \int_0^1 (t^6 - t^4) 2t \, dt + (2t^2)(t^3) 3t \, dt \]

\[ = \int_0^1 (2t^4 - 2t^3 + 3t^2) \, dt \]

\[ = \int_0^1 (2t^4 - 2t^3 + 3t^2) \, dt \]

\[ = \left[ \frac{1}{3} t^5 - \frac{1}{2} t^4 + \frac{3}{2} t^3 \right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}. \]
\[ \int_{C} f \, ds := \int_{a}^{b} f(\vec{r}(t)) \, \|\vec{r}'(t)\| \, dt \]  

(\ast)

\[ \int_{C} f \, dx := \int_{a}^{b} f(\vec{r}(t)) \, \vec{r}'(t) \, dt \]  

(\ast\ast)

**Properties:**

1. \( \int_{C} f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds \)

   \[ \int_{C} f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds \]  

   for both (\ast) and (\ast\ast).

2. \( \int_{-C} f \, ds = \int_{C} f \, ds \) but \( \int_{-C} f \, dx = -\int_{C} f \, dx \)

**Note:** \( \sqrt{x(t)^2 + ... + x_n(t)^2} \) doesn't change sign, but \( x_i'(t) \) does, when you reverse the path.

In 2 or 3 variables, then one written:

\[ \int_{C} f \, dx, \int_{C} f \, dy, \int_{C} f \, dz. \]

**Ex 4:** Evaluate the line integral

\[ \int_{C} (x^2 - y^2) \, dx + 2xy \, dy \]

Combining them like this is the typical notation along the curve parametrized by \( \vec{r}(t) = (t^3, t^2) \).

\[ dx = d(t^3) = 3t^2 \, dt, \quad dy = d(t^2) = 2t \, dt \]

\[ \int_{C} (x^2 - y^2) \, dx + 2xy \, dy \]

\[ = \int_{0}^{1} [(t^3)^2 - (t^2)^2] \, 3t^2 \, dt + 2t^3(t^2) \, 2t \, dt \]

\[ = \int_{0}^{1} (2t^5 - 2t^4 + 6t^3) \, dt \]

\[ = \left[ \frac{1}{3}t^6 - \frac{1}{2}t^5 + 2t^4 \right]_{0}^{1} = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} . \]
\[
\int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt
\]  
\((*)\)

\[
\int_C f \, dx := \int_a^b f(\mathbf{r}(t)) \mathbf{r}'(t) \, dt
\]  
\((***)\)

**Properties:**

1. \(C = C_1 + C_2 \quad \Rightarrow \quad \int_C f = \int_{C_1} f + \int_{C_2} f\)

2. \(\int_C f \, ds = \int_C f \, ds, \quad \text{but} \quad \int_C f \, dx = -\int_{-C} f \, dx;\)

   \(h/c\) \(\sqrt{x(t)^2 + y(t)^2}\) doesn't change sign, but \(x'(t) dy\) does, when you reverse the path.

3. Independence of parametrization of a given oriented curve \(C\)

In 2 or 3 variables then can be written

\[
\int_C f \, dx, \quad \int_C f \, dy, \quad \int_C f \, dz.
\]

**Ex 4/ Evaluate the line integral**

\[
\int_C (x^2 - y^2) \, dx + 2xy \, dy
\]

Combining them like this is typical notation along the curve parametrized by \(\mathbf{r}(t) = (t^2, t^3)\) for \(t \in [0, 1]\).

\[
dx = d(t^2) = 2t \, dt, \quad dy = d(t^3) = 3t^2 \, dt.
\]

\[
\Rightarrow \quad \int_C \ldots = \int_0^1 \left( (t^4) - (t^3) \right) 2t \, dt + (t^3)(3t^2) 3t \, dt
\]

\[
= \int_0^1 (2t^5 - 2t^4 + 3t^5) \, dt
\]

\[
= \left[ \frac{1}{3} t^6 - \frac{1}{2} t^5 + \frac{3}{4} t^4 \right]_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.
\]
\[ \int_{C} f \, ds := \int_{a}^{b} f(\mathbf{r}(t)) \left| \mathbf{r}'(t) \right| \, dt \] (\ast)

\[ \int_{C} f \, dx_{i} := \int_{a}^{b} f(\mathbf{r}(t)) \mathbf{r}'_{i}(t) \, dt \] (**)

**Properties:**

1. \[ C = C_1 + C_2 \quad \Rightarrow \quad \int_{C} f = \int_{C_1} f + \int_{C_2} f \] for both (\ast) and (**).

2. \[ \int_{-C} f \, ds = \int_{C} f \, ds \] but \[ \int_{-C} f \, dx_{i} = -\int_{C} f \, dx_{i} \]

3. Independence of parameterization of a given oriented curve \( C \)

What about independence of path \( C \) from a given \( A \) to a given \( B \)?
\[ \int_C f \, ds := \int_a^b f(\mathbf{r}(t)) \left\| \mathbf{r}'(t) \right\| \, dt \quad (\star) \]
\[ \int_C f \, dx_i := \int_a^b f(\mathbf{r}(t)) \mathbf{r}'_i(t) \, dt \quad (***) \]

**Properties:**

1. \[ \oint_{C_1 + C_2} = \oint_{C_1} + \oint_{C_2} \]
   for both (\star) and (***)

2. \[ \int_C f \, ds = \int_C f \, ds, \quad \text{but} \quad \int_C f \, dx_i = -\int_C f \, dx_i \]
   if \( \sqrt{x_1'^2 + \ldots + x_n'^2} \) doesn't change sign, but \( x_i'(t) \) does, when you reverse the path

3. Independence of parametrization of a given oriented curve \( C \)

What about independence of path \( C \) from a given \( A \) to a given \( B \)?

Clearly, this does not hold for (\star) or (***) in general. Say \( f \equiv 1 \) on the green disk \( D \) and \( f \equiv 0 \) outside the blue one, and \( f \) is \( \geq 0 \) and continuous everywhere. Then, \( \int_C f \, ds, \int_C f \, dx_i, \int_C f \, dx_i \) are all \( \geq 0 \) while \( \int_C f \, ds, \int_C f \, dy, \int_C f \, dy \) are all \( = 0 \).
We'll have to wait until next week to discover a situation when "independence of path" does hold.
Now let's go back to the drawing board. Suppose we have a particle moving in the plane along a curve $C$, in some sort of force field $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$. Assume continuous $= (P(x,y), Q(x,y))$. 
Now let's go back to the drawing board. Suppose we have a particle moving in the plane along a curve \( C \), in some sort of force field \( \vec{F}(x,y) = P(x,y)i + Q(x,y)j \). Assume \( \vec{F} \) is continuous.

What is the total work done by \( \vec{F} \) on the particle?
Now let's go back to the drawing board. Suppose we have a particle moving in the plane along a curve $C$, in some sort of force field $\mathbf{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$. Assume continuity.

What is the total work done by $\mathbf{F}$ on the particle? Again we partition $C$ and estimate

$$\Delta W_i \approx \mathbf{F}(x^{(i)}, y^{(i)}) \cdot \mathbf{T}(x^{(i)}, y^{(i)})(\Delta s);$$

$$W \approx \sum_{i=1}^{n} \Delta W_i;$$
Now let's go back to the drawing board. Suppose we have a particle moving in the plane along a curve \( C \), in some sort of force field \( \vec{F}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j} \). Assume continuous.

What is the total work done by \( \vec{F} \) on the particle? Again we partition \( C \)

\[ C = \{ (x^m, y^m) \} \]

and estimate

\[ (\Delta W)_i \approx \vec{F}(x^{(i)}, y^{(i)}) \cdot \hat{T}(x^{(i)}, y^{(i)})(\Delta s); \]

\[ W \approx \sum_{i=1}^{n} (\Delta W)_i; \]

Taking the limit as the partition becomes finer and \( n \to \infty \), we get

\[ W = \int_C \vec{F}(x, y) \cdot \hat{t}(x, y) \, ds \]

\[ = \int_{a}^{b} \vec{F}(\hat{r}(s)) \cdot \hat{T}(\hat{r}(s)) \| \hat{T}(s) \| \, ds \]

parametrizing

\[ C \] by \( \hat{r}(s) \),

\( s \in [a, b] \).
Now let's go back to the drawing board. Suppose we have a particle moving in the plane along a curve $C$, in some sort of force field $\vec{F}(x,y) = P(x,y)\hat{i} + Q(x,y)\hat{j}$. Assume continuous force field $\vec{F}(x,y)$.

What is the total work done by $\vec{F}$ on the particle? Again we partition $C$ and estimate

$$\Delta W_i \approx \vec{F}(x^{(n)}_i, y^{(n)}_i) \cdot \hat{T}(x^{(n)}_i, y^{(n)}_i)(\Delta s);$$

$$W \approx \sum_{i} \Delta W_i;$$

Taking the limit as the partition becomes finer and $n \to \infty$, we get

$$W = \int_C \vec{F}(x,y) \cdot \hat{T}(x,y) \, ds$$

$$= \int_{a}^{b} \vec{F}(\hat{r}(t)) \cdot \hat{T}(\hat{r}(t)) \, ||\vec{r}'(t)|| \, dt$$

parametrizing $C$ by $\hat{r}(t)$, $t \in [a,b]$.

$$= \int_{a}^{b} \vec{F}(\hat{r}(t)) \cdot \hat{r}'(t) \, dt.$$
Now let's go back to the drawing board. Suppose we have a particle moving in the plane along a curve $C$, in some sort of force field $\vec{F}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j}$. Assume continuity.

What is the total work done by $\vec{F}$ on the particle? Again we partition $C$ by $t_i, t_i^{(n)}$, and estimate $\Delta W_i \approx \vec{F}(x_i^{(n)}, y_i^{(n)}) \cdot \vec{T}(x_i^{(n)}, y_i^{(n)})(\Delta s)$.

Total work $W \approx \sum_{i=1}^{n} \Delta W_i$.

Taking the limit as the partition becomes finer and $n \to \infty$, we get

$$W = \int_C \vec{F}(x, y) \cdot \vec{T}(x, y) \, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{T}(\vec{r}(t)) \, \left\| \vec{r}'(t) \right\| \, dt$$

parametrizing $C$ by $\vec{r}(t), t \in [a, b]$,

$$\approx \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{T}(t) \, dt$$

Definition: Let $F: \mathcal{S} \to \mathbb{R}^n$ be $C^0$, with $\mathcal{S}$ containing $C$; and $\vec{r}: [a, b] \to \mathbb{R}^n$ a $C^2$ parameterization of $C$. The line integral of $F$ along $C$ is

$$\int_C \vec{F} \cdot d\vec{s} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{T}(t) \, dt$$
We can also write this as
\[(i) \int_C \mathbf{F} \cdot d\mathbf{s}\]

Taking the limit as the partition becomes finer and \(n \to \infty\), we get
\[W = \int_C \mathbf{F}(x,y) \cdot \mathbf{T}(x,y) \, ds = \int_0^1 F(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|| \mathbf{r}'(t) ||} \, dt\]

parametrizing \(C \subset \mathbb{R}^2\), \(t \in [a,b]\),
\[= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt .\]

**Definition:** Let \( \mathbf{F} : \mathcal{S} \to \mathbb{R}^n \) be \(C^0\), with \( \mathcal{S} \) containing \( C \); and \( \mathbf{r} : [a,b] \to \mathbb{R}^n \) a \( C^1 \) parametrization of \( C \). The line integral of \( \mathbf{F} \) along \( C \) is,
\[\int_C \mathbf{F} \cdot d\mathbf{s} := \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt .\]
We can also write this as

\[(i) \quad \int_{C} \vec{F} \cdot d\vec{s} \quad \text{or} \quad \int_{C} \vec{F} = (f_{1}, \ldots, f_{n})\]

\[(ii) \quad \int_{a}^{b} \sum_{i=1}^{n} f_{i}(\vec{r}(t)) \cdot \frac{d\vec{r}(t)}{dt} \, dt = \int_{C} f_{1} \, dx_{1} + \cdots + f_{n} \, dx_{n}\]

Taking the limit as the partition becomes finer and \(n \to \infty\), we get

\[W = \int_{C} \vec{F}(x, y) \cdot \vec{T}(x, y) \, ds\]

\[= \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \, ||\vec{r}'(t)|| \, dt\]

parametrizing \(C \subset \mathbb{R}^{2}\), such as \(\vec{r}(t)\), \(t \in [a, b]\)

\[= \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt\] (green)

**Definition:** Let \(\vec{F} : S \to \mathbb{R}^{n}\) be \(C^{0}\), with \(S\) containing \(C\); and \(\vec{r} : [a, b] \to \mathbb{R}^{n}\) a \(C^{1}\) parametrization of \(C\). The line integral of \(\vec{F}\) along \(C\) is

\[\int_{C} \vec{F} \cdot d\vec{s} := \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt\] (red)
We can also write this as

(i) \[ \int_C \hat{F} \cdot d\mathbf{s} \quad \text{or} \quad \hat{F} = (f_1, \ldots, f_n) \]

(ii) \[ \sum_{i=1}^n f_i(x_i(t)) \Delta x_i \Delta t = \sum f_i \, dx_i + \cdots + f_n \, dx_n \]

from either of which one may deduce that

- \( \int_C \hat{F} \cdot d\mathbf{s} = -\int_C \hat{F} \cdot d\mathbf{s} \) (why?)
- \( \int_C \hat{F} \cdot d\mathbf{s} + \int_C \hat{F} \cdot d\mathbf{s} = \int_C \hat{F} \cdot d\mathbf{s} \)

not to mention
- \( \int_C (a\hat{F} + b\hat{G}) \cdot d\mathbf{s} = a\int_C \hat{F} \cdot d\mathbf{s} + b\int_C \hat{G} \cdot d\mathbf{s} \).

Taking the limit as the partition becomes finer and \( n \to \infty \), we get

\[ W = \int_C \hat{F}(x,y) \cdot \hat{T}(x,y) \, ds \]

\[ = \int_a^b \hat{F}(\hat{r}(t)) \cdot \frac{\hat{r}'(t)}{||\hat{r}'(t)||} ||\hat{r}'(t)|| \, dt \]

parametrizing \( C \) by \( \hat{r}(t) \),
\( C \in C_1 \), \( \epsilon = 0 \)

\[ = \int_a^b \int_C \hat{F}(\hat{r}(t)) \cdot \frac{\hat{r}'(t)}{||\hat{r}'(t)||} ||\hat{r}'(t)|| \, dt. \]

**Definition:** Let \( \hat{F} : S \to \mathbb{R}^n \) be \( \mathcal{C}^0 \), with \( S \) containing \( C \); and \( \hat{r} : [a,b] \to \mathbb{R}^n \) a \( \mathcal{C}^1 \) parametrization of \( C \). The line integral of \( \hat{F} \) along \( C \) is

\[ \int_C \hat{F} \cdot d\mathbf{s} := \int_a^b \hat{F}(\hat{r}(t)) \cdot \hat{r}'(t) \, dt. \]
We can also write this as

\( i. \) \[ \int \vec{F} \cdot \vec{d}s \quad \text{or} \quad \vec{F} = (F_1, \ldots, F_n) \]

\( \text{or} \]

\( ii. \) \[ \int_a^b \sum_{i=1}^n f_i(t_i(x)) \cdot \vec{r}(t_i(x)) \, dt_i = \int_a^b f_1(x) \, dx + \ldots + f_n(x) \, dx \]

from either of which one may deduce that:

\( \bullet \) \[ \int -e_i \vec{F} \cdot \vec{d}s = -\int \vec{F} \cdot \vec{d}s \quad \text{(why?)} \]

\( \bullet \) \[ \int \vec{F} \cdot \vec{d}s + \int \vec{F} \cdot \vec{d}s = \int \vec{F} \cdot \vec{d}s \]

not to mention:

\( \bullet \) \[ \int (a\vec{F} + b\vec{G}) \cdot \vec{d}s = a\int \vec{F} \cdot \vec{d}s + b\int \vec{G} \cdot \vec{d}s. \]

Independence of the choice of parametrization \( \vec{r} \) of \( C \) follows from \( (i) \), and also by imitating the argument from before:

Taking the limit as the partition becomes fine and \( n \to \infty \), we get:

\[ W = \int \vec{F}(x,y) \cdot \vec{r}(x,y) \, ds \]

\[ \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}(\vec{r}(t)) \, I_{R^n} \, dt \]

parametrizing \( C \) by \( \vec{r}(t) \),

\[ C_{\vec{r}(t), \vec{r}(0)=a, \vec{r}(1)=b} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt. \]

**Definition:** Let \( \vec{F} : \mathcal{S} \to \mathbb{R}^n \) be \( \mathcal{C}^0 \), with \( \mathcal{S} \) containing \( C \); and \( \vec{r} : [a,b] \to \mathbb{R}^n \) a \( \mathcal{C}^1 \) parametrization of \( C \). The line integral of \( \vec{F} \) along \( C \) is:

\[ \int_{C} \vec{F} \cdot d\vec{s} := \int_{a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt. \]
We can also write this as

\[(i) \quad \int \mathbf{F} \cdot d\mathbf{s} \quad \text{or} \quad \int \mathbf{F} = (f_1, \ldots, f_n) \]

\[(ii) \quad \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{r}(t) \, dt = \int_a^b f_1 \, dx_1 + \cdots + f_n \, dx_n \]

from either of which one may deduce that

- \( \int \mathbf{F} \cdot d\mathbf{s} = -\int \mathbf{F} \cdot d\mathbf{s} \quad \text{(why?)} \)
- \( \int \mathbf{F}_1 \cdot d\mathbf{s} + \int \mathbf{F}_2 \cdot d\mathbf{s} = \int (\mathbf{F}_1 + \mathbf{F}_2) \cdot d\mathbf{s} \)
- \( \int_a^b (a\mathbf{F} + b\mathbf{G}) \cdot d\mathbf{s} = a\int \mathbf{F} \cdot d\mathbf{s} + b\int \mathbf{G} \cdot d\mathbf{s} \).

Independence of the choice of parametrization \( \mathbf{r} \) of \( C \) follows from \((i)\), and also by imitating the argument from before:

\[\begin{align*}
[\mathbf{c}, \mathbf{l}] & \xrightarrow{g} [\mathbf{c}(u), \mathbf{l}(u)] \xrightarrow{\mathbf{r}(u)} \mathbb{R}^n \\
\mathbf{r}(u) &= \mathbf{F}(g(u)) \\
\int_C \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{n}(u) \, du &= \int_a^b \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) \, du \\
&= \int_a^b \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) \, du \\
& \iff \mathbf{t} = g(u) \quad \mathbf{d}u = g'(u) \, du
\end{align*}\]
\[
\begin{align*}
[u] & \to [c, b] \to R^n \\
\hat{r}(u) = \hat{r}(g(u)) \\
\int \hat{F}(\hat{r}(u), \hat{r}(u)) \, du &= \int \hat{F}(\hat{r}(g(u)), \hat{r}(g(u)) g'(u) \, du \\
&= \int_a^b \hat{F}(\hat{r}(t)), \hat{r}'(t) \, dt \\
t = g(u) \\
dt = g'(u) \, du
\end{align*}
\]

Let's record the formula once more:
\[
\int \hat{F} \cdot \hat{r}' \, dt := \int_a^b \hat{F}(\hat{r}(t)) \cdot \hat{r}'(t) \, dt \\
= \int_c f_1 \, dx + \ldots + f_n \, dx
\]
Ex 5] Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{s} \) on the two paths from \( A \) to \( B \) indicated:

\[
\begin{align*}
C_1 & \quad \rightarrow \\
C_2 & \quad (1,1) = B \\
C_3 & : y = x^2 \\
A = (0,0)
\end{align*}
\]

when \( \mathbf{F}(x,y) = (xy^2, x^2-y^2) \).

\[
\begin{align*}
\mathbf{r} & : [c, d] \rightarrow \mathbb{R}^n \\
\mathbf{r}(u) & = \mathbf{F}(g(u)) \\
\mathbf{r}'(u) & = \mathbf{F}'(g(u)) \cdot g'(u) \\
\int_C \mathbf{F} \cdot d\mathbf{s} & = \int_C \mathbf{r}'(u) \cdot \mathbf{F}(g(u)) \, du \\
& = \int_c^d \mathbf{F}(\mathbf{r}(u)) \cdot \mathbf{r}'(u) \, du \\
& = \int_c^d \mathbf{r}'(u) \cdot \mathbf{F}(\mathbf{r}(u)) \, du
\end{align*}
\]

\[
\begin{align*}
t & = g(u) \\
du & = g'(u) \, du
\end{align*}
\]

\[
\begin{align*}
\text{Let's record the formula once more:} \\
\int_C \mathbf{F} \cdot d\mathbf{s} & = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\
& = \int_C f_1 \, dx + \cdots + f_n \, dy
\end{align*}
\]
Ex 5] Evaluate $\int_C \vec{F} \cdot d\vec{r}$ on the two paths from A to B indicated:

\[ C_1 \quad C_2 \quad C_3 : y = x^2 \]

\[ A = (0,0) \]

When $\vec{F}(x,y) = (xy^2, x^2 - y^2)$.

\[
\int_{C_3} \vec{F} \cdot d\vec{r} = \int_{C_3} xy^2 \, dx + (x^2 - y^2) \, dy
\]

\[ \vec{F}(1,1) = (x, y) \]

\[
\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 x(t^4) \, dt + (t^2 - (t^4))^2 \, dt = \int_0^1 (t^2 + 2t^3 - t^5) \, dt
\]

\[ = [\frac{1}{2}t^4 - \frac{1}{6}t^6]_0^1 = \frac{1}{3}. \]

Let's record the formula once more:

\[
\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

\[
= \int_C f_1 \, dx_1 + \ldots + f_n \, dx_n
\]
Ex 5] Evaluate $\int_C \vec F \cdot d\vec s$ on the
two paths from A to B indicated:

- $C_1$ from $(0,0)$ to $(1,1)$
- $C_2$ from $(1,1)$ to $(0,0)$
- $C_3$ from $(0,0)$ to $(1,1)$, $y = x^2$

When $\vec F(x,y) = (xy^2, x^2 - y^2)$.

\[
\int_{C_1} \vec F \cdot d\vec s = \int_0^1 xy^2 \, dx + (x^2 - y^2) \, dy
\]

\[
\int_{C_2} \vec F \cdot d\vec s = \int_0^1 t(t^2 - 1) \, dt + (t^2 - t^2) \, dt^2
\]

\[
\int_{C_3} \vec F \cdot d\vec s = \int_0^1 (t^3 + 2t^3 - t^3) \, dt
\]

\[
= \left[ \frac{1}{2} t^4 - \frac{1}{6} t^6 \right]_0^1 = \frac{1}{3}.
\]

\[
\int_C \vec F \cdot d\vec s = \int_{C_1} \vec F \cdot d\vec s + \int_{C_2} \vec F \cdot d\vec s
\]

\[
\vec F(t) = (t, 1-t)
\]

\[
\vec r(t) = (0,t) \Rightarrow d\vec s = (dt, 0)
\]

Let's record the formula once more:

\[
\int_C \vec F \cdot d\vec s = \int_a^b \vec F(\vec r(t)) \cdot \vec r'(t) \, dt
\]

\[
= \int_C f_1 \, dx + \ldots + f_n \, dx
\]
Ex 5: Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) on the two paths from A to B indicated:

\[ A = (0,0) \]

\[ (1,1) = B \]

\[ C_1 \]

\[ C_2 \]

\[ C_3 : y = x^2 \]

\[ \vec{F}(x, y) = (xy^2, x^2 - y^2) \]

\[ \begin{align*}
\int_{C_3} \vec{F} \cdot d\vec{r} &= \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} \\
\vec{F}(x, y) &= (xy^2, x^2 - y^2) \\
&= \int_0^1 (0^2 - t^2) \, dt + \int_0^1 t(2t) \, dt \\
&= \left[ -\frac{t^3}{3} \right]_0^1 + \left[ \frac{t^2}{2} \right]_0^1 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.
\end{align*} \]

So independence of path from A to B holds.

Let's record the famous once more:

\[ \int_C \vec{F} \cdot d\vec{r} := \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \\
= \int_C f_1 \, dx + \ldots + f_n \, dx_n \]
Application:

\[ \int_{c_1 \cup c_2} \mathbf{F} \cdot d\mathbf{s} = \int_{c_1} \mathbf{F} \cdot d\mathbf{s} + \int_{c_2} \mathbf{F} \cdot d\mathbf{s} \]

\[ \text{Use } \mathbf{r}(t) = (\cos t, \sin t) \Rightarrow d\mathbf{s} = (\cos t, \sin t) \, dt \]
\[ \text{Note: } \mathbf{r}(t) = (0, t) \Rightarrow d\mathbf{s} = (0, dt) \]

\[ = \int_0^1 (6t^2 - t^3) \, dt + \int_0^1 t(t^2)^2 \, dt \]
\[ = \left[ -\frac{t^3}{3} \right]_0^1 + \left[ \frac{t^2}{2} \right]_0^1 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6} \]

So independence of path from A to B holds.

---

Let's record the formula once more:

\[ \int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \]

\[ = \int_{c_1} f_1 \, dx_1 + \cdots + f_n \, dx_n \]
Application: If \( \vec{r}(t) \) is the motion of an object in a force field \( \vec{F}(\vec{r}) \), then Newton's 2nd law says
\[
\vec{F}(\vec{r}(t)) = m \vec{r}''(t) = m \vec{a}(t).
\]

\[
\begin{align*}
\int_{C_1+C_2} \vec{F} \cdot d\vec{x} &= \int_{C_1} \vec{F} \cdot d\vec{x} + \int_{C_2} \vec{F} \cdot d\vec{x} \\
&= \int_0^1 (2^2-t^2) \, dt + \int_0^1 t(2)^2 \, dt \\
&= -\frac{t^3}{3} \bigg|_0^1 + \frac{t^2}{2} \bigg|_0^1 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.
\end{align*}
\]

So independence of path from \( A \) to \( B \) holds.

Let's record the formula once more:
\[
\int_C \vec{F} \cdot d\vec{x} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \\
= \int_C f_1 \, dx_1 + \ldots + f_n \, dx_n
\]
Application: If $\vec{r}(t)$ is the motion of an object in a force field $\vec{F}(\vec{x})$, then Newton's 2nd law says

$$\vec{F}(\vec{r}(t)) = m \vec{v}''(t) = m \ddot{\vec{v}}(t).$$

The work done by the force on the object as it moves along $C$ is then

$$W = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_a^b m \ddot{\vec{v}}(t) \cdot \vec{v}(t) \, dt$$

$$= \frac{1}{2} m \Delta [\vec{v}(t)] \bigg|_a^b$$

$$= \int_C \vec{F} \cdot d\vec{x} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_a^b f_1 \, dx_1 + \cdots + f_n \, dx_n$$

So independence of path from $A$ to $B$.

Let's record the formula once more:

$$\int_C \vec{F} \cdot d\vec{x} := \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_a^b f_1 \, dx_1 + \cdots + f_n \, dx_n$$
Application: If \( \vec{v}(t) \) is the motion of an object in a force field \( \vec{F}(\vec{r}) \), then Newton's 2nd law says

\[
\vec{F}(\vec{r}(t)) = m \vec{v}''(t) = m \ddot{\vec{v}}(t).
\]

The work done by the force on the object as it moves along \( C \) is therefore

\[
W = \int_C \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt
= \int_a^b m \ddot{\vec{v}}(t) \cdot \vec{v}(t) \, dt
= \frac{1}{2} m \left[ \dot{\vec{v}}(t) \cdot \dot{\vec{v}}(t) \right]_a^b
= \frac{1}{2} m \left[ \vec{v}(b)^2 - \vec{v}(a)^2 \right].
\]

\[
\text{FTC} = \frac{1}{2} m \left[ \vec{v}(b)^2 - \vec{v}(a)^2 \right]
= \Delta KE,
\]

the change in the kinetic energy of the object.

\[
\begin{align*}
\int_C \vec{F} \cdot d\vec{s} &= \int_C \vec{F} \cdot d\vec{s} + \int_{C_1} \vec{F} \cdot d\vec{s} \\
&= \int_C \vec{F} \cdot d\vec{s} + \int_{C_1} \vec{F} \cdot d\vec{s} \\
&= \int \left( 0^2 - x^2 \right) \, dt + \int t(2)^2 \, dt \\
&= \left. -\frac{x^3}{3} \right|_1^0 + \left. \frac{t^2}{2} \right|_1^0
= -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.
\end{align*}
\]

So, independence of path from \( A \) to \( B \) days.

Let's record the famous once more:

\[
\int_C \vec{F} \cdot d\vec{s} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{v}(t) \, dt
= \int_C F_1 \, dx + \ldots + F_n \, dx
\]
Zam Cookies

200 g flour
1/2 T cinnamon
1/2 T ginger
1/2 t each cloves, cardamom, nutmeg, coriander, cayenne (optional)
1 t baking soda
100 g salted butter (or unsalted butter + 1/2 t salt)
100 g molasses or golden syrup
100 g brown sugar
1/2 t boiling water
granulated sugar

(except for the granulated sugar)

Mix together all dry ingredients except for the granulated sugar.
Cream butter, brown sugar, molasses until fluffy; add the tiny bit of boiling water, then the dry ingredients, mix together.
Shape the mixture into small balls, roll them in the granulated sugar, place on parchment, flatten slightly.

Bake in 350°F preheated oven for 15 minutes.

Let them cool before eating.

(you have had these before...)