Lecture 35: The FTC for line integrals

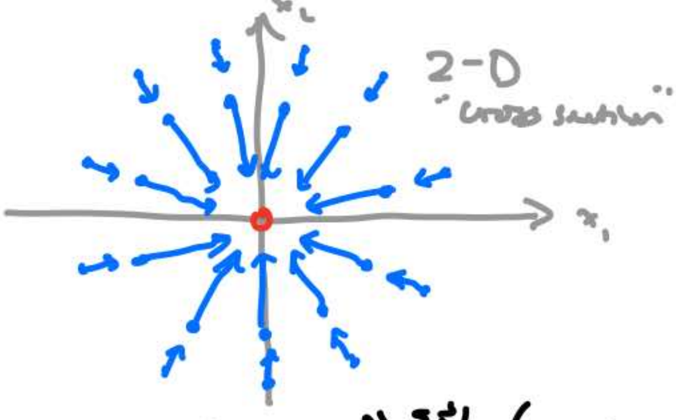
According to Newton, the magnitude of force of gravitational attraction between objects of mass M & m is $\frac{GMm}{d^2}$, where G is a universal constant and d is the distance between the objects.

According to Ptolemy, the Earth ($= M$) sits at the origin.

Let's describe the force field it exerts on an object of mass m: writing $\vec{x} = (x_1, x_2, x_3)$ for its position and $r := \|\vec{x}\|$ for its
or x, y, z
distance from $\vec{0}$, the force has magnitude

$$\|\vec{F}(\vec{x})\| = \frac{GMm}{r^2} = \frac{GMm}{\|\vec{x}\|^2}$$

The direction is toward the origin, i.e. in the direction of the unit vector $-\frac{\vec{x}}{\|\vec{x}\|}$. Hence



$$\vec{F}(\vec{x}) = \frac{GMm}{\|\vec{x}\|^2}\left(\frac{-\vec{x}}{\|\vec{x}\|}\right)$$

$$= -GMm\, r^{-3}\vec{x}$$

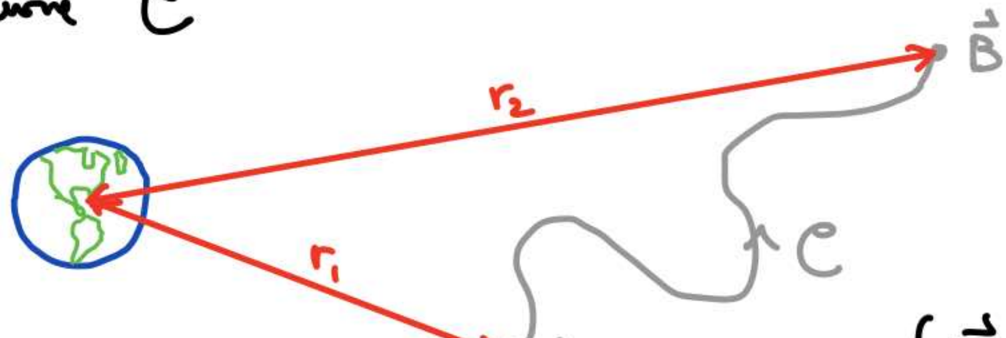
This is a vector field defined on $\mathcal{S} = \mathbb{R}^3 \setminus \{\vec{0}\}$ (complement of the origin).

Notice that $\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\sqrt{\sum_j x_j^2} = \frac{2x_i}{2\sqrt{\sum x_j}} = \frac{x_i}{r}$

$\Rightarrow \vec{\nabla} r^k = k r^{k-1}\vec{\nabla} r = k r^{k-2}\vec{x}$. Defining $\varphi: \mathcal{S} \to \mathbb{R}$ by $\varphi(\vec{x}) = GMm\, r^{-1}$, we therefore have $\vec{\nabla}\varphi = \vec{F}$: that is, φ is a potential function for $\vec{F}$.

We define the <u>potential energy</u> of the object moving in the force field to be $PE := -\varphi(\vec{x})$. (This is only well-defined up to a constant, so it is <u>changes</u> in PE that are meaningful.)

Recall that the work done by $\vec{F}$ as the object moves along the curve $\mathcal{C}$



is $W = \int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \underbrace{\Delta KE}_{\text{change in kinetic energy}}$

It would be nice if this was

$$-\Delta PE = \varphi(\vec{B}) - \varphi(\vec{A}) = GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right),$$

so that energy is conserved. This is precisely the content of today's first theorem.

$\boxed{\text{SETTING}}$

We shall assume today that $\mathcal{S} \subseteq \mathbb{R}^n$ is a <u>connected</u> open set: for all $\vec{A}, \vec{B} \in \mathcal{S}$, there exists a piecewise smooth curve $\mathcal{C} \subseteq \mathcal{S}$ starting at $\vec{A}$ and ending at $\vec{B}$ (written $\underline{\partial[C] = \vec{B} - \vec{A}}$).

Equivalently, $\mathcal{S}$ is <u>not</u> a disjoint union of ≥ 2 open sets:

We shall do the 2^{nd} fundamental theorem first; it says that for a gradient field $\vec{F} = \vec{\nabla}\varphi$ on $\mathcal{S}$, we indeed have

$$\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = \varphi(\vec{B}) - \varphi(\vec{A}).$$

FTC II : Given $f : \mathcal{S} \to \mathbb{R}$ differentiable, with $\vec{\nabla} f$ continuous on $\mathcal{S}$, and $\mathcal{C} \subset \mathcal{S}$ piecewise smooth† oriented curve with $\partial[\mathcal{C}] = \vec{B} - \vec{A}$,

$$\boxed{\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = f(\vec{B}) - f(\vec{A}).}$$

Proof : For $\mathcal{C}$ smooth, $\displaystyle\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$$= \lim_{\substack{t_0 \to a^+ \\ t_1 \to b^-}} \int_{t_0}^{t_1} \underbrace{\vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)}_{g'(t),\text{ where}}\, dt = \lim_{\substack{t_0 \to a^+ \\ t_1 \to b^-}} \left(g(t_1) - g(t_0) \right)$$

$g(t) = f(\vec{r}(t))$ is continuous on $[a,b]$

$$= g(b) - g(a) = f(\vec{B}) - f(\vec{A}).$$

For $\mathcal{C}$ only piecewise smooth,

$$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = \sum_{i=1}^{n} \int_{\mathcal{C}_i} \vec{\nabla} f \cdot d\vec{r} = \sum_{i=1}^{n} \left(f(\vec{A}_i) - f(\vec{A}_{i-1}) \right) = f(\vec{B}) - f(\vec{A}). \qquad \square$$

Corollary : If $\mathcal{C}$ is a closed path, i.e. $\vec{B} = \vec{A}$ (written $\partial[\mathcal{C}] = 0$), then $\displaystyle\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = 0$.

† Here "$\mathcal{C}$ is smooth" is meant in Apostol's (weak) sense: the parametrization $\vec{r} : [a,b] \to \mathcal{S}$ is assumed to have continuous derivative $\vec{r}\,'$ on (a,b).

Ex/ Recall that $\vec{F}(\vec{r}) = \frac{-c\vec{x}}{\|\vec{x}\|^3} = \vec{\nabla} f$, $f = \frac{c}{\|\vec{x}\|}$.

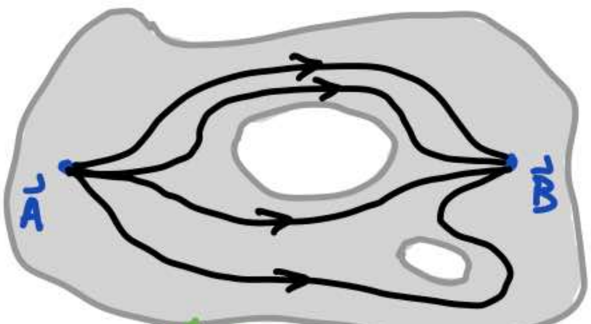
Let C be the line segment connecting $\vec{A} = (0,3,0)$ to $\vec{B} = (4,3,0)$. We compute

$$\int_C \vec{F}\cdot d\vec{r} = f(4,3,0) - f(0,3,0) = \frac{c}{5} - \frac{c}{3} = -\frac{2c}{15}. \quad //$$

Next remember that FTC I said that $\frac{d}{dx}\int_{x_0}^{x} F(t)\,dt = F(x)$.

Here we want to say $\vec{\nabla}\int_{\vec{x}_0}^{\vec{x}} \vec{F}\cdot d\vec{r} = \vec{F}(\vec{x})$, but without an assumption, $\int_{\vec{x}_0}^{\vec{x}} \vec{F}\cdot d\vec{r}$ makes no sense!

Let $\vec{F} : \mathcal{S} \to \mathbb{R}^n$ be continuous. We shall say that "$\int \vec{F}\cdot d\vec{r}$ is independent of path on $\mathcal{S}$" if for each $\vec{A}, \vec{B} \in \mathcal{S}$, $\int_C \vec{F}\cdot d\vec{r}$ gives the same number regardless of what (piecewise smooth) C with $\partial[C] = \vec{B} - \vec{A}$ is chosen.



Equivalently, we shall say that the vector field $\vec{F}$ is conservative.

Write $\int_{\vec{A}}^{\vec{B}} \vec{F}\cdot d\vec{r}$ for the line integral over any of these paths.

So $\int_{\vec{A}}^{\vec{B}} \vec{F}\cdot d\vec{r}$ makes sense for every $\vec{A}, \vec{B} \in \mathcal{S}$ if and only if $\vec{F}$ is conservative (on $\mathcal{S}$).

includes that $\vec{F}$ is continuous

FTC I : Given $\vec{F} : \mathcal{S} \to \mathbb{R}$ conservative and $\vec{A} \in \mathcal{S}$, $\varphi(\vec{x}) := \int_{\vec{A}}^{\vec{x}} \vec{F}\cdot d\vec{r}$ is well-defined and $\vec{\nabla}\varphi = \vec{F}$.

<u>Proof</u>: Choose $\vec{x} \in \mathcal{S}$, $r > 0$ s.t. $B(\vec{x}; r) \subset \mathcal{S}$, and $\hat{u}$ a unit vector: for $0 \le h < r$,

$$\varphi(\vec{x} + h\hat{u}) - \varphi(\vec{x}) = \int_{\vec{A}}^{\vec{x}+h\vec{u}} \vec{F} \cdot d\vec{r} - \int_{\vec{A}}^{\vec{x}} \vec{F} \cdot d\vec{r}$$

$$= \int_{\vec{x}}^{\vec{x}+h\vec{u}} \vec{F} \cdot d\vec{r}$$

$\vec{r}(t) = \vec{x} + t\hat{u}$
$t \in [0, h]$

$$= \int_0^h F(\vec{x} + t\vec{u}) \cdot \hat{u}\, dt$$

$\hat{u} = \hat{e}_k$

$$\Longrightarrow \varphi(\vec{x} + h\hat{e}_k) - \varphi(\vec{x}) = \int_0^h \vec{F}(\vec{x} + t\vec{e}_k) \cdot \hat{e}_k\, dt$$

$F = (f_1, \dots, f_n)$

$$= \int_0^h f_k(\vec{x} + t\hat{e}_k)\, dt =: g(h)$$

note $g(0) = 0$

1-variable FTC

$$\Longrightarrow g'(h) = f_k(\vec{x} + h\hat{e}_k)$$

$$\Longrightarrow \frac{\partial \varphi}{\partial x_k}(\vec{x}) = \lim_{h \to 0} \frac{\varphi(\vec{x} + h\hat{e}_k) - \varphi(\vec{x})}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= g'(0) = f_k(\vec{x})$$

$$\Longrightarrow \vec{\nabla}\varphi = (f_1, \dots, f_n) = \vec{F}. \qquad \square$$

So FTC I says that <u>conservative fields have a potential function</u>. We'll talk about how to construct this potential function computationally next time.

A SIMPLE APPLICATION TO DEs

A differential of the form

$$(*) \qquad P(x,y)\,dx + Q(x,y)\,dy$$

is called <u>exact</u> if $P = \frac{\partial \varphi}{\partial x}$ and $Q = \frac{\partial \varphi}{\partial y}$ for some function $\varphi(x,y)$.

Writing $(*) = 0$ gives the DE

$$(**) \qquad P + Q\frac{dy}{dx} = 0.$$

<u>Proposition</u>: Assume $(*)$ is exact. Then the solutions

$$y = Y(x) \quad \text{of} \quad (**)$$

are precisely the functions whose graphs are level curves of φ.

<u>Proof</u>: Writing $g(x) = \varphi(x, Y(x))$, we have

$$g'(x) = \frac{\partial \varphi}{\partial x}(x, Y(x)) + \frac{\partial \varphi}{\partial y}(x, Y(x))\,Y'(x) = P(x, Y(x)) + Q(x, Y(x))\frac{dY}{dx}$$

and so Y solves $(**)$ $\Leftrightarrow$ g is constant

$\Leftrightarrow$ $(x, Y(x))$ traces out level curve. $\square$

Ex/ We can use this to solve equations of the form

$\frac{dy}{dx} = G(x,y)$. For example,

$$\frac{dy}{dx} = \frac{-y}{2x} \rightsquigarrow \underbrace{y\,dx + 2x\,dy}_{\text{not exact — so ~~doesn't work~~?}} = 0$$

NOT SO FAST!

Multiply by y $\rightsquigarrow$ $\underbrace{y^2}_{\frac{\partial}{\partial x}xy^2}dx + \underbrace{2xy}_{\frac{\partial}{\partial y}xy^2}dy = 0$ (so $\varphi = xy^2$)

Level sets are $xy^2 = C$

$\Rightarrow$ solutions are $Y(x) = \sqrt{\frac{C}{x}}$.

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