Lecture 35

The Fundamental Theorems of Calculus for Line Integrals
According to Newton, the magnitude of force of gravitational attraction between objects of mass $M$ and $m$ is $\frac{G M m}{d^2}$, where $G$ is a universal constant and $d$ is the distance between the objects.
According to Newton, the magnitude of force of gravitational attraction between objects of mass M and m is \( GMm/d^2 \), where G is a universal constant and d is the distance between the objects. According to Ptolemy, the Earth (\( =M \)) sits at the origin. Let's describe the force field it exerts on an object of mass m:
According to Newton, the magnitude of force of gravitational attraction between objects of mass \(M\) and \(m\) is \(G \frac{Mm}{d^2}\), where \(G\) is a universal constant and \(d\) the distance between the objects.

According to Ptolemy, the Earth (=M) sits at the origin. Let's describe the force field it exerts on an object of mass \(m\): writing \(\vec{x} = (x_1, x_2, x_3)\) or \((x, y, z)\) for its position and \(r := \|\vec{x}\|\) for its distance from \(\hat{0}\), the force has magnitude \(\|\vec{F}(\vec{x})\| = G\frac{Mm}{r^2} = G\frac{Mm}{\|\vec{x}\|^2}\).
According to Newton, the magnitude of force of gravitational attraction between objects of mass \( M \) and \( m \) is \( \frac{GMm}{d^2} \), where \( G \) is a universal constant and \( d \) is the distance between the objects.

According to Ptolemy, the Earth \((=M)\) sits at the origin. Let's describe the force field it exerts on an object of mass \( m \): writing \( \mathbf{x} = (x_1, x_2, x_3) \) or \( (x, y, z) \) for its position and \( r := \|\mathbf{x}\| \) for its distance from \( \mathbf{0} \), the force has magnitude \( \|\mathbf{F}(\mathbf{x})\| = \frac{GMm}{r^2} = \frac{GMm}{\|\mathbf{x}\|^2} \).

The direction is toward the origin, i.e., in the direction of the unit vector \( -\frac{\mathbf{x}}{\|\mathbf{x}\|} \).
According to Newton, the magnitude of force of gravitational attraction between objects of mass \( M \) and \( m \) is \( \frac{GMm}{d^2} \), where \( G \) is a universal constant and \( d \) is the distance between the objects.

According to Ptolemy, the Earth (=M) sits at the origin. Let's describe the force field it exerts on an object of mass \( m \): writing \( \mathbf{x} = (x_1, x_2, x_3) \) or \( (x, y, z) \) for its position and \( r := ||\mathbf{x}|| \) for its distance from \( \hat{0} \), the force has magnitude \( |\mathbf{F}(\mathbf{x})| = \frac{GMm}{r^2} = \frac{GMm}{||\mathbf{x}||^2} \).

The direction is toward the origin, i.e. in the direction of the unit vector \( \frac{\mathbf{x}}{||\mathbf{x}||} \).
According to Newton, the magnitude of force of gravitational attraction between objects of mass $M$ and $m$ is $\frac{GMm}{d^2}$, where $G$ is a universal constant and $d$ the distance between the objects. According to Ptolemy, the Earth ($=M$) sits at the origin. Let's describe the force field it exerts on an object of mass $m$: writing $\mathbf{x} = (x_1, x_2, x_3)$ or $(\mathbf{x}, y, z)$ for its position and $r := ||\mathbf{x}||$ for its distance from $0$, the force has magnitude $|F(x)| = \frac{GMm}{r^2} = \frac{GMm}{||\mathbf{x}||^2}$. The direction is toward the origin, i.e. in the direction of the unit vector $\frac{-\mathbf{x}}{||\mathbf{x}||}$. 

$$\Rightarrow F(x) = \frac{GMm}{||\mathbf{x}||^2} \left( \frac{-\mathbf{x}}{||\mathbf{x}||} \right) = -GMm \frac{\mathbf{x}}{||\mathbf{x}||^3}.$$ 
This vector field is defined on the component of the origin $\mathbf{r} = (x, y, z)$.
According to Newton, the magnitude of force of gravitational attraction between objects of mass \( M \) and \( m \) is \( \frac{GMm}{d^2} \), where \( G \) is a universal constant and \( d \) is the distance between the objects.

According to Ptolemy, the Earth \((=M)\) sits at the origin. Let's describe the force field it exerts on an object of mass \( m \): writing \( \mathbf{x} = (x_1, x_2, x_3) \) or \((x, y, z)\) for its position and \( r := ||\mathbf{x}|| \) for its distance from \( \hat{0} \), the force has magnitude \( ||\mathbf{F}(\mathbf{x})|| = \frac{GMm}{r^2} = \frac{GMm}{||\mathbf{x}||^2} \).

The direction is toward the origin, i.e., in the direction of the unit vector \( -\frac{\mathbf{x}}{||\mathbf{x}||} \).

\[ \Rightarrow \mathbf{F}(\mathbf{x}) = \frac{GMm}{||\mathbf{x}||^2} \left( -\frac{\mathbf{x}}{||\mathbf{x}||} \right) = -GMm \frac{\mathbf{x}}{r^3} \]

This vector field is defined on the complement of the origin \( B = \mathbb{R}^3 \setminus \{0\} \).

Notice that \( \frac{\partial}{\partial x_i} = \frac{1}{\sqrt{2} x_j} \mathbf{x}_j = \frac{2x_i}{2\sqrt{2}x_j^2} = \frac{x_i}{r} \).
According to Newton, the magnitude of force of gravitational attraction between objects of mass \( M \) and \( m \) is \( \frac{GMm}{d^2} \), where \( G \) is a universal constant and \( d \) is the distance between the objects.

According to Ptolemy, the Earth (=M) sits at the origin. Let's describe the force field it exerts on an object of mass \( m \): writing \( \vec{x} = (x_1, x_2, x_3) \) or \( (x, y, z) \) for its position and \( r := \| \vec{x} \| \) for its distance from \( \vec{0} \), the force has magnitude \( \| \vec{F}(\vec{x}) \| = \frac{GMm}{r^2} = \frac{GMm}{\| \vec{x} \|^2} \).

The direction is toward the origin, i.e. in the direction of the unit vector \( -\frac{\vec{x}}{\| \vec{x} \|} \).

\[ \Rightarrow \vec{F}(\vec{x}) = \frac{GMm}{\| \vec{x} \|^2} \left( -\frac{\vec{x}}{\| \vec{x} \|} \right) = \frac{-GMm}{\| \vec{x} \|^3} \vec{x} \]

This vector field is defined on the component of the origin \( B = \mathbb{R}^3 \setminus \{\vec{0}\} \). Notice that \( \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_j} \sqrt{x_j^2} = \frac{2x_i}{2\sqrt{x_j^2}} = \frac{x_i}{r} \)

\[ \Rightarrow \nabla r^{-k} = k r^{-k-1} \nabla r = kr^{-k-2} \vec{x} \]
According to Newton, the magnitude of force of gravitational attraction between objects of mass \( M \) and \( m \) is \( \frac{GMm}{d^2} \), where \( G \) is a universal constant and \( d \) is the distance between the objects.

According to Ptolemy, the Earth \((=M)\) sits at the origin. Let's describe the force field it exerts on an object of mass \( m \): writing \( \vec{x} = (x_1, x_2, x_3) \) or \((x, y, z)\) for its position and \( r := ||\vec{x}|| \) for its distance from \( \hat{0} \), the force has magnitude \( ||\vec{F}(\vec{x})|| = \frac{GMm}{r^2} = \frac{GMm}{||\vec{x}||^2} \).

The direction is toward the origin, i.e. in the direction of the unit vector \( \frac{-\vec{x}}{||\vec{x}||} \).

\[
\Rightarrow \vec{F}(\vec{x}) = \frac{GMm}{||\vec{x}||^2} \left( \frac{-\vec{x}}{||\vec{x}||} \right) = -GMm \frac{r^{-3}}{||\vec{x}||^2} \frac{-\vec{x}}{||\vec{x}||}.
\]

This vector field is defined on the component of the origin \( S = \mathbb{R}^3 \setminus \{0\} \).

Notice that \( \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \sqrt{\sum_j x_j^2} = \frac{x_i}{2\sqrt{x_i^2}} = \frac{x_i}{r} \)

\[
\Rightarrow \nabla r^{-k} = k r^{-k-1} \nabla r = k r^{-k-2} \vec{x}.
\]

Define \( \varphi : S \to \mathbb{R} \) by \( \varphi(\vec{x}) := GMm r^{-1} \),

we therefore have \( \nabla \varphi = \frac{\vec{x}}{r} \).
According to Newton, the magnitude of force of gravitational attraction between objects of mass \( M \) and \( m \) is \( \frac{GMm}{d^2} \), where \( G \) is a universal constant and \( d \) the distance between the objects.

According to Ptolemy, the Earth \((=M)\) sits at the origin. Let's describe the force field it exerts on an object of mass \( m \): writing \( \mathbf{x} = (x, y, z) \) for its position and \( r := \| \mathbf{x} \| \) for its distance from \( \mathbf{0} \), the force has magnitude

\[
\| \mathbf{F}(x) \| = \frac{GMm}{r^2} = \frac{GMm}{\| \mathbf{x} \|^2}.
\]

The direction is toward the origin, i.e., in the direction of the unit vector \( \frac{x}{\| \mathbf{x} \|} \).

\[
\Rightarrow \mathbf{F}(x) = \frac{GMm}{\| x \|^2} \left( \frac{x}{\| x \|} \right) = -GMm \frac{x}{\| x \|^3}.
\]

This vector field is defined on the component of the origin \( \mathbb{S} = \mathbb{R}^3 \setminus \{ \mathbf{0} \} \).

Notice that

\[
\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_j} (\sqrt{x_j^2}) = \frac{x_j}{2\sqrt{x_j^2}} = \frac{x_j}{r}
\]

\[
\Rightarrow \nabla r^{-1} = kr^{-1} \nabla r = kr^{-2} \frac{\mathbf{x}}{\| \mathbf{x} \|}.
\]

Defining \( \varphi : \mathbb{S} \to \mathbb{R} \) by \( \varphi(x) := GMm r^{-1} \), we therefore have \( \nabla \varphi = \mathbf{F} \); that is, \( \varphi \) is a potential function for \( \mathbf{F} \).
According to Newton, the magnitude of force of gravitational attraction between objects of mass \( M \) and \( m \) is \( \frac{GMm}{d^2} \), where \( G \) is a universal constant and \( d \) is the distance between the objects. According to Ptolemy, the Earth \((=M)\) sits at the origin. Let's describe the force field it exerts on an object of mass \( m \): writing \( \mathbf{x} = (x_1, x_2, x_3) \) or \((x, y, z)\) for its position and \( r := \|\mathbf{x}\| \) for its distance from \( \hat{0} \), the force has magnitude \( \|\mathbf{F}(\mathbf{x})\| = \frac{GMm}{r^2} = \frac{GMm}{\|\mathbf{x}\|^2} \).

The direction is toward the origin, i.e., in the direction of the unit vector \( -\frac{\mathbf{x}}{\|\mathbf{x}\|} \).

\[ \Rightarrow \mathbf{F}(\mathbf{x}) = \frac{GMm}{\|\mathbf{x}\|^2} \left(-\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) = -GMm \frac{1}{\|\mathbf{x}\|^3} \mathbf{x}. \]

This vector field is defined on the component of the origin \( \mathcal{B} = \mathbb{R}^3 \setminus \{0\} \).

Notice that \( \frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i} \sqrt{x_j^2} = \frac{x_j}{\sqrt{x_j^2}} = \frac{x_j}{r} \) \( \Rightarrow \nabla r^{-k} = kr^{k-1} \nabla r = kr^{k-2} \frac{\mathbf{x}}{r} \).

Defining \( \varphi : \mathcal{A} \to \mathbb{R} \) by \( \varphi(\mathbf{x}) := \frac{GMm}{r} \), we therefore have \( \nabla \varphi = \frac{\mathbf{F}}{r} \); that is, \( \varphi \) is a potential function for \( \mathbf{F} \).

We define the potential energy of the object moving in the force field to be \( PE := -\varphi(\mathbf{x}) \).
(This is only well-defined up to a constant.)

\[ F(\mathbf{x}) = \frac{GMm}{|\mathbf{x}|^2} \left( \frac{-\hat{x}}{|\mathbf{x}|^2} \right) = -GMm r^{-3} \hat{x}. \]

This vector field is defined on the component of the origin \( \mathbb{S} = \mathbb{R}^3 \setminus \{0\} \).

Notice that \( \frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i} \sqrt{\sum_j x_j^2} = \frac{2x_i}{2\sqrt{x_j^2}} = \frac{x_i}{r} \),

\[ \Rightarrow \nabla r^{-k} = kr^{k-1} \nabla r = kr^{k-2} \hat{x}. \]

Defining \( \varphi : \mathbb{S} \to \mathbb{R} \) by \( \varphi(\mathbf{x}) := GMm r^{-1} \), we therefore have \( \nabla \varphi = F \): that is, \( \varphi \) is a potential function for \( F \). We define the potential energy of the object moving in the force field to be \( PE := -\varphi(\mathbf{x}) \).
(This is only well-defined up to a constant.)
Recall that the work done by $F$ as the object moves along the curve $C$
\[ W = \int_C \mathbf{F} \cdot d\mathbf{s} = \Delta KE. \]

\[ \Rightarrow F(x) = \frac{GMm}{|x|^2} \left( -\frac{x}{|x|} \right) = -GMmr^{-3} \hat{x}. \]

This vector field is defined on the component of the origin, $\mathbb{B} = \mathbb{R}^3 \setminus \{0\}$.

Notice that $\frac{\partial r}{\partial x_j} = \frac{\partial}{\partial x_j} \sqrt{x_i^2 + x_j^2} = \frac{x_i x_j}{2\sqrt{x_i^2 + x_j^2}} = \frac{x_i}{r}$
\[ \Rightarrow \nabla r^{-k} = kr^{k-1} \nabla r = kr^{k-2} \hat{x}. \]

Defining $\varphi : \mathbb{B} \to \mathbb{R}$ by $\varphi(x) := GMmr^{-1}$,
we therefore have \[ \nabla \varphi = \hat{x} \] that is, $\varphi$ is a potential function for $F$.

We define the potential energy of the object moving in the force field $F$ to be
\[ PE := -\varphi(x). \]
(This is only well-defined up to a constant.)

Recall that the work done by $\mathbf{F}$ as the object moves along the curve $C$

$$W = \int_C \mathbf{F} \cdot \, d\mathbf{r} = \Delta K\text{E}.$$  
change in kinetic energy

It would be nice if this was

$$-\Delta P\text{E} = \varphi(\mathbf{B}) - \varphi(\mathbf{A})$$

$$= GMm \left( \frac{1}{r_2} - \frac{1}{r_1} \right),$$

so that energy is conserved.

$$\Rightarrow \mathbf{F}(\mathbf{x}) = \frac{GMm}{\|\mathbf{x}\|^2} \left( \frac{\mathbf{x}}{\|\mathbf{x}\|} \right) = -GMm \mathbf{r}^{-3} \hat{\mathbf{r}}.$$

This vector field is defined on the component of the origin $\mathbb{B} = \mathbb{R}^3 \setminus \{0\}$.

Notice that \( \frac{\partial r}{\partial x_i} = \frac{\partial \sqrt{x_j^2}}{\partial x_j} = \frac{2x_j}{2\sqrt{x_j^2}} = \frac{x_j}{r} \)

\( \Rightarrow \nabla r^{-k} = kr^{-k-1} \nabla r = kr^{-k-2} \mathbf{r} \).

Defining $\varphi : \mathbb{B} \to \mathbb{R}$ by $\varphi(\mathbf{x}) = GMm \mathbf{r}^{-1}$, we therefore have $\nabla \varphi = \mathbf{F}$; that is, $\varphi$ is a potential function for $\mathbf{F}$.

We define the potential energy of the object moving in the force field to be $P\text{E} := -\varphi(\mathbf{x})$. 

(This is only well-defined up to a constant.)

Recall that the work done by \( \mathbf{F} \) as the object moves along the curve \( \mathcal{C} \)

is 
\[
W = \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \Delta K E.
\]

\text{change in kinetic energy}

It would be nice if this was 
\[
-\Delta PE = \varphi (\mathbf{B}) - \varphi (\mathbf{A})
\]
\[
= G m M \left( \frac{1}{r_2} - \frac{1}{r_1} \right),
\]
so that energy is conserved. This is precisely the content of today's 1st theorem.

\[
\Rightarrow \varphi (\mathbf{x}) = \frac{GMm}{|\mathbf{x}|^2} \left( \frac{\mathbf{x}}{|\mathbf{x}|} \right) = -GMm r^{-3} \frac{x}{r}.
\]

This vector field is derived on the component of the origin, \( \mathcal{B} = (\mathbb{R}^3 \backslash \{0\}) \).

Notice that 
\[
\frac{\partial}{\partial x_i} \sqrt{x_j x_j} = \frac{2 x_i}{2 \sqrt{x_j x_j}} = \frac{x_i}{r}
\]
\[
\Rightarrow \nabla r^{-k} = k r^{-k-1} \nabla r = kr^{-k-2} \frac{x}{r}.
\]

Defining \( \varphi : \mathcal{B} \rightarrow \mathbb{R} \) by \( \varphi (\mathbf{x}) := GMm r^{-1} \),
we therefore have \( \nabla \varphi = \mathbf{F} \), that is, \( \varphi \) is a potential function for \( \mathbf{F} \).

We define the potential energy of the object moving in the force field to be 
\[
PE := -\varphi (\mathbf{x}).
\]
(This is only well-defined up to a constant.)
Recall that the work done by $F$ as the object moves along the curve $C$

is $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \Delta KE$.

It would be nice if this was

$-\Delta PE = \varphi(B) - \varphi(A)$

$= GM \left( \frac{1}{r_2} - \frac{1}{r_1} \right)$,

so that energy is conserved. This is precisely the content of today's 1st law theorem.
(This is only well-defined up to a constant.)
Recall that the work done by \( \mathbf{F} \) as the object moves along the curve \( C \)

\[
W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \Delta \text{KE}.
\]

It would be nice if this was

\[
-\Delta \text{PE} = \varphi (\mathbf{B}) - \varphi (\mathbf{A}) = GmM \left( \frac{1}{r_2} - \frac{1}{r_1} \right),
\]

so that energy is conserved. This is precisely the content of today's 1st Law theorem.
(This is only well-defined up to a constant.)
Recall that the work done by \( \mathbf{F} \) as the object moves along the curve \( C \)
\[ W = \int_C \mathbf{F} \cdot d\mathbf{r} = \Delta \text{KE}. \]

It would be nice if this was
\[ -\Delta PE = \varphi(\mathbf{B}) - \varphi(\mathbf{A}) = GmM \left( \frac{1}{r_2} - \frac{1}{r_1} \right), \]
so that energy is conserved. This is precisely the content of today's 1st law theorem.

**SETTING**

We shall assume today that \( \mathcal{D} \subseteq \mathbb{R}^n \) is a connected open set: for all \( \mathbf{A}, \mathbf{B} \in \mathcal{D} \), there exists a piecewise smooth curve \( C \) starting at \( \mathbf{A} \) and ending at \( \mathbf{B} \) (written \( \mathcal{J}[C] = \mathbf{B} - \mathbf{A} \)). Equivalently, \( \mathcal{D} \) is not a disjoint union of \( \geq 2 \) open sets.
(This is only well-defined up to a constant.)

Recall that the work done by $\mathbf{F}$ as the object moves along the curve $C$

$$\mathbf{F} = \nabla \rho$$

is $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \Delta \text{KE}$.

Setting

We shall assume today that $S \subseteq \mathbb{R}^n$ is a connected open set: for all $\mathbf{A}, \mathbf{B} \in S$, there exists a piecewise smooth curve $C$ starting at $\mathbf{A}$ and ending at $\mathbf{B}$ (written $\mathbf{B} - \mathbf{A}$). Equivalently, $S$ is not a disjoint union of $\geq 2$ open sets.

It would be nice if this were

$$-\Delta \text{PE} = \rho(\mathbf{B}) - \rho(\mathbf{A})$$

$$= G \frac{m M}{\left( \frac{1}{r_2} - \frac{1}{r_1} \right)}$$

so that energy is conserved. This is precisely the content of today's 1st law.

We shall do the 2nd fundamental theorem first; it says that for a gradient field $\mathbf{F} = \nabla \rho$ on $S$, we indeed have $\int_C \mathbf{F} \cdot d\mathbf{r} = \rho(\mathbf{B}) - \rho(\mathbf{A})$. 
\textbf{FTC II}: Given \( f: S \rightarrow \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( S \), and \( C = \overline{A, B} \) piecewise smooth oriented curve with \( \delta C = \overline{B - A} \),
\[
\int_C \nabla f \cdot \, ds = f(B) - f(A).
\]

\textbf{SETTING}:

We shall assume today that \( S \subseteq \mathbb{R}^n \) is a connected open set: for all \( \overline{A, B} \subseteq S \), there exists a piecewise smooth curve \( C \) starting at \( \overline{A} \) and ending at \( \overline{B} \) (written \( \delta [C] = \overline{B - A} \)). Equivalently, \( S \) is not a disjoint union of \( \geq 2 \) open sets:

We shall do the 2\textsuperscript{nd} fundamental theorem first; it says that for a gradient field \( F = \nabla \varphi \) on \( S \), we indeed have \( \int_C F \cdot \, ds = \varphi(B) - \varphi(A) \).
FTC II: Given $f: S \rightarrow \mathbb{R}$ differentiable, with $\frac{df}{dx}$ continuous on $S$, and $C = \gamma$ piecewise smooth oriented curve with $\gamma[C] = \vec{B} - \vec{A}$,

$$\int_{\gamma} \frac{df}{dx} \cdot d\gamma = f(\vec{B}) - f(\vec{A}).$$

N.B.: Here “$C$ is smooth” is meant in the Apostol (weak) sense: The parametrization $\vec{r}: (a, b) \rightarrow \mathbb{R}^n$ is assumed to have continuous derivative $\vec{r}'$ on $(a, b)$.

**SETTING**

We shall assume today that $S \subseteq \mathbb{R}^n$ is a connected open set: for all $\vec{A}, \vec{B} \in S$, there exists a piecewise smooth curve $C$ starting at $\vec{A}$ and ending at $\vec{B}$ (written $\gamma[C] = \vec{B} - \vec{A}$). Equivalently, $S$ is not a disjoint union of $\geq 2$ open sets:

We shall do the 2nd fundamental theorem first; it says that for a gradient field $\vec{F} = \nabla \varphi$ on $S$, we indeed have $\int_{C} \vec{F} \cdot d\gamma = \varphi(\vec{B}) - \varphi(\vec{A}).$
**FTC II:** Given \( f: \mathbb{R} \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathcal{A} \), and \( \mathcal{C} = \mathcal{A} \) piecewise smooth oriented curve with \([\mathcal{C}] = \mathcal{B} - \mathcal{A}\),

\[
\int_{\mathcal{C}} \nabla f \cdot d\mathbf{s} = f(\mathcal{B}) - f(\mathcal{A})
\]

**Setting**

We shall assume today that \( \mathcal{A} \subseteq \mathbb{R}^n \) is a connected open set: for all \( \mathbf{A}, \mathbf{B} \in \mathcal{A} \), there exists a piecewise smooth curve \( \mathcal{C} \) starting at \( \mathbf{A} \) and ending at \( \mathbf{B} \) (written \([\mathcal{C}] = \mathbf{B} - \mathbf{A}\)). Equivalently, \( \mathcal{A} \) is not a disjoint union of \( \geq 2 \) open sets.

We shall do the 2nd fundamental theorem first; it says that for a gradient field \( \mathbf{F} = \nabla \varphi \) on \( \mathcal{A} \), we indeed have

\[
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \varphi(\mathcal{B}) - \varphi(\mathbf{A})
\]
FTC II: Given \( f: S \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( S \), and \( C = \beta - \alpha \) piecewise smooth oriented curve with \( \Delta[C] = \beta - \alpha \),

\[ \int_C \nabla f \cdot d\mathbf{	au} = f(\beta) - f(\alpha). \]

N.B.: Here "C is smooth" is meant in the Apostol (weak) sense: The parametrization \( \mathbf{r}: (a, b) \to \mathbb{R}^n \) is assumed to have continuous derivative \( \mathbf{r}' \) on \((a, b)\).

Proof: \( \int_C \nabla f \cdot d\mathbf{	au} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \)

\[ = \lim_{t_0 \to a^+} \int_{t_0}^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \]

\[ \text{SETTING} \]

We shall assume today that \( S \subseteq \mathbb{R}^n \) is a connected open set: for all \( \mathbf{A}, \mathbf{B} \in S \), there exists a piecewise smooth curve \( C \) starting at \( \mathbf{A} \) and ending at \( \mathbf{B} \) (written \( \Delta[C] = \beta - \alpha \)). Equivalently, \( S \) is not a disjoint union of \( \geq 2 \) open sets:

We shall do the 2-nd fundamental theorem first; it says that for a gradient field \( \mathbf{F} = \nabla \varphi \) on \( S \), we indeed have \( \int_C \mathbf{F} \cdot d\mathbf{	au} = \varphi(\beta) - \varphi(\alpha) \).
FTC II: Given \( f: \mathbb{R} \to \mathbb{R} \) differentiable, with \( \frac{df}{dt} \) continuous on \( \mathbb{R} \), and \( C = \mathbb{R} \) piecewise smooth oriented curve with \( C = \mathbb{R}^- \cdot \mathbb{R}^+ \), then
\[
\int_C \frac{df}{dt} \cdot ds = f(\mathbb{R}^+) - f(\mathbb{R}^-).
\]

N.B.: Here "\( C \) is smooth" is meant in the Apostol (weak) sense: The parameterization \( \mathbf{r}: (a,b) \to \mathbb{R}^n \) is assumed to have continuous derivative \( \mathbf{r}' \) on \( (a,b) \).

Proof: \[
\int_C \frac{df}{dt} \cdot ds = \int_a^b \frac{df}{dt}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.
\]
\[
= \lim_{t_0 \to a^+} \int_{t_0}^b \frac{df}{dt}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,
\]
where \( g(t) = f(\mathbf{r}(t)) \).

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**SETTING**

We shall assume today that \( \mathcal{S} \subseteq \mathbb{R}^n \) is a connected open set: for all \( \mathbf{A}, \mathbf{B} \in \mathcal{S} \), then there exists a piecewise smooth curve \( C \) starting at \( \mathbf{A} \) and ending at \( \mathbf{B} \) (written \( \Delta [C] = \mathbf{B} - \mathbf{A} \)). Equivalently, \( \mathcal{S} \) is not a disjoint union of \( \geq 2 \) open sets:

We shall do the 2nd fundamental theorem first: it says that for a gradient field \( \mathbf{F} = \nabla \psi \) on \( \mathcal{S} \), we indeed have \[
\int_C \mathbf{F} \cdot ds = \psi(\mathbf{B}) - \psi(\mathbf{A}).
\]
FTC II: Given \( f : \mathbb{R} \to \mathbb{R} \) differentiable with \( \nabla f \) continuous on \( \mathbb{R} \), and \( C = \mathbb{R} \) piecewise smooth oriented curve with \( \nabla f \cdot \mathbf{r}'(t) \, dt \) finite,

\[
\int_C \nabla f \cdot \mathbf{r}' \, dt = f(B) - f(A).
\]

N.B.: Here "\( C \) is smooth" is meant in the Apostol (weak) sense: The parametrization \( \mathbf{r} : (a,b) \to \mathbb{R}^n \) is assumed to have continuous derivative \( \mathbf{r}' \) on \( (a,b) \).

Proof: \( \int_C \nabla f \cdot \mathbf{r}' \, dt = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \)

\[
= \lim_{t \to +} \int_{t_0}^{t_1} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt,
\]

where \( g(t) = f(\mathbf{r}(t)) \), and

\[
\lim_{t \to +} (g(t_1) - g(t_0)) = \lim_{t \to +} \int_{t_0}^{t_1} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt.
\]

Setting

We shall assume today that \( \mathcal{B} \subseteq \mathbb{R}^n \) is a connected open set: for all \( \mathbf{A}, \mathbf{B} \in \mathcal{B} \), there exists a piecewise smooth curve \( C \) starting at \( \mathbf{A} \) and ending at \( \mathbf{B} \) (written \( \mathcal{D}[C] = \mathbf{B} - \mathbf{A} \)). Equivalently, \( \mathcal{B} \) is not a disjoint union of \( \geq 2 \) open sets.

We shall do the 2nd fundamental theorem first: it says that for a gradient field \( \mathbf{F} = \nabla \varphi \) on \( \mathcal{B} \), we indeed have \( \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A) \).
FTC II: Given \( f : S \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( S \), and \( C = S \) piecewise smooth oriented curve with \( \Delta[C] = \mathbf{B} - \mathbf{A} \),

\[
\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A})
\]

N.B.: Here "C is smooth" is meant in the Apostol (weak) sense: The parametrization \( \mathbf{r} : (a, b) \to \mathbb{R}^n \) is assumed to have continuous derivative \( \mathbf{r}' \) on \((a, b)\).

Proof:

\[
\int_C \nabla f \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \lim_{t \to t_0^+} \int_{t_0^-}^{t_0^+} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \lim_{t \to t_0^+} \left( g(t) - g(t_0^-) \right)
\]

\[
= g(b) - g(a)
\]

Setting

We shall assume today that \( S \subseteq \mathbb{R}^n \) is a connected open set: for all \( \mathbf{A}, \mathbf{B} \in S \), there exists a piecewise smooth curve \( C \) starting at \( \mathbf{A} \) and ending at \( \mathbf{B} \) (written \( \Delta[C] = \mathbf{B} - \mathbf{A} \)). Equivalently, \( S \) is not a disjoint union of \( \geq 2 \) open sets:

We shall do the 2nd fundamental theorem first; it says that for a gradient field \( \mathbf{F} = \nabla \varphi \) on \( S \), we indeed have \( \int_S \mathbf{F} \cdot d\mathbf{S} = \varphi(\mathbf{B}) - \varphi(\mathbf{A}) \).
FTC II: Given \( f: \mathbb{R} \to \mathbb{R} \) differentiable, with \( \frac{\partial f}{\partial x} \) continuous on \( \mathbb{R} \), and \( C = \gamma \) a piecewise smooth oriented curve with \( \gamma[C] = \hat{B} - \hat{A} \),

\[
\int_{\gamma} \frac{\partial f}{\partial x} \cdot \mathrm{d}s = f(\hat{B}) - f(\hat{A}).
\]

N.B.: Here “\( C \) is smooth” is meant in the Apostol (weak) sense: The parametrization \( \gamma: (a,b) \to \mathbb{R}^n \) is assumed to have continuous derivative \( \gamma' \) on \((a,b)\).

Proof: \[
\int_{\gamma} \frac{\partial f}{\partial x} \cdot \mathrm{d}s = \int_{a}^{b} \frac{\partial f}{\partial x}(\gamma(t)) \cdot \gamma'(t) \mathrm{d}t
\]

\[
= \lim_{t_0 \to a^+} \int_{t_0}^{b} \frac{\partial f}{\partial x}(\gamma(t)) \cdot \gamma'(t) \mathrm{d}t,
\]

where \( g(t) = f(\gamma(t)) \), \( g(t) \) is continuous on \( [a, b] \).

\[
= \lim_{t_0 \to a^+} (g(b) - g(a))
\]

\[
= g(b) - g(a) = f(\hat{B}) - f(\hat{A}).
\]
FTC II: Given $f: \mathbb{R} \rightarrow \mathbb{R}$ differentiable, with $\nabla f$ continuous on $\mathbb{R}$, and $C = \gamma$ piecewise smooth oriented curve with $\gamma[C] = \vec{B} - \vec{A}$,

$$\int_{C} \nabla f \cdot d\gamma = f(\vec{B}) - f(\vec{A}).$$

N.B.: Here "$C$ is smooth" is meant in the Apostol (weak) sense: The parametrization $\vec{r}: (a,b) \rightarrow \mathbb{R}^n$ is assumed to have continuous derivative $\vec{r}'$ on $(a,b)$.

Proof:

$$\int_{C} \nabla f \cdot d\gamma = \int_{a}^{b} \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \lim_{t_{0} \to t_{+}} \int_{t_{0}}^{t} \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \lim_{t_{0} \to t_{+}} (g(t_{+}) - g(t_{0}))$$

where $g(t) = f(\vec{r}(t))$

is continuous on $[a, b]$

$$= g(b) - g(a) = f(\vec{B}) - f(\vec{A}).$$

If $C$ is only piecewise smooth, then

$$\int_{C} \nabla f \cdot d\gamma = \sum_{i=1}^{n} \int_{C_{i}} \nabla f \cdot d\gamma$$

$$= \sum_{i=1}^{n} (f(\vec{A}_{i}) - f(\vec{A}_{i-1}))$$

(to use Rieman sum) $= f(\vec{B}) - f(\vec{A})$. □
**FTC II:** Given \( f : \mathbb{R} \rightarrow \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathbb{R} \), and \( C = \gamma \) piecewise smooth oriented curve with \( \mathbb{Z}[C] = \bar{B} - \bar{A} \),

\[
\int_C \nabla f \cdot \, d\mathbf{s} = f(\bar{B}) - f(\bar{A}).
\]

N.B. : Here “\( C \) is smooth” is meant in the Apostol (weak) sense : The parametrization \( \bar{\varphi} : [a,b] \rightarrow \mathbb{R}^n \) is assumed to have continuous derivative \( \bar{\varphi}' \) on \( [a,b] \).

Proof : \( \int_C \nabla f \cdot \, d\mathbf{s} = \int_a^b \nabla f(\bar{\varphi}(t)) \cdot \bar{\varphi}'(t) \, dt \)

\[
= \lim_{t_0 \to t^+} \left( g(t_0) - g(t_0^-) \right) = f(\bar{B}) - f(\bar{A}).
\]

If \( C \) is only piecewise smooth, then

\[
\int_C \nabla f \cdot \, d\mathbf{s} = \sum_{i=1}^{n} \int_{C_i} \nabla f \cdot \, d\mathbf{s} = \sum_{i=1}^{n} (f(\bar{A}_i) - f(\bar{A}_{i-1}))
\]

+ Use continuity of \( g(t) = f(\bar{\varphi}(t)) \) on \( [a,b] \)

Corollary : If \( C \) is a closed path, i.e. \( \bar{B} = \bar{A} \) (written \( \mathbb{Z}[C] = 0 \)), then

\[
\int_C \nabla f \cdot \, d\mathbf{s} = 0.
\]
FTC II: Given \( f: \mathbb{R} \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( B \), and \( C = B \) piecewise smooth oriented curve with \( \partial C = \vec{B} - \vec{A} \),

\[
\int_C \nabla f \cdot \, d\vec{s} = f(\vec{B}) - f(\vec{A})
\]

Ex: Recall that \( \vec{F}(\vec{r}) = \frac{-\vec{C}}{11 x^2 y^2} = \nabla f \), where \( f(\vec{r}) = \frac{C}{11} \). Let \( C \) be the line segment connecting \( \vec{A} = (0,3,0) \) to \( \vec{B} = (4,3,0) \).

If \( C \) is only piecewise smooth, then

\[
\int_C \nabla f \cdot \, d\vec{s} = \sum_{i=1}^{\infty} \int_{C_i} \nabla f \cdot \, d\vec{s}
= \sum_{i=1}^{\infty} (f(\vec{A}_i) - f(\vec{A}_{i-1}))
\]

tending to

\[
f(\vec{B}) - f(\vec{A})
\]

Corollary: If \( C \) is a closed path, i.e., \( \vec{B} - \vec{A} \) (written \( \partial C = 0 \)), then

\[
\int_C \nabla f \cdot \, d\vec{s} = 0.
\]
**FTC II**: Given $f: \mathbb{R} \rightarrow \mathbb{R}$ differentiable, with $\nabla f$ continuous on $\mathbb{R}$, and $C = \overline{AB}$ a piecewise smooth, oriented curve with $\partial C = \overline{B-A}$, we have

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A).$$

**Example**: Recall that $F(x) = \frac{-x^2}{2l^2} = \nabla f$, where $f(x) = \frac{x}{ll^2}$. Let $C$ be the line segment connecting $A = (0,3,0)$ to $B = (4,3,0)$. We compute

$$\int_C \nabla f \cdot d\vec{r} = f(4,3,0) - f(0,3,0) = \frac{4}{5} - \frac{0}{3} = \frac{4}{5}.$$ 

If $C$ is only piecewise smooth, then

$$\int_C \nabla f \cdot d\vec{r} = \sum_{i=0}^{n} \int_{C_i} \nabla f \cdot d\vec{r} = \sum_{i=1}^{n} \sum_{i=1}^{n} (f(\overrightarrow{A_i}) - f(\overrightarrow{A_{i-1}})),$$

to which sum $f(B) - f(A)$. \quad \Box$

**Corollary**: If $C$ is a closed path, i.e., $B = A$ (written $\partial C = 0$), then

$$\int_C \nabla f \cdot d\vec{r} = 0.$$
FTC II: Given $f: \mathbb{R} \to \mathbb{R}$ differentiable, with $\nabla f$ continuous on $\mathbb{R}$, and $C$ a piecewise smooth oriented curve with $\delta(C) = \vec{B} - \vec{A}$,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\vec{B}) - f(\vec{A}).$$

Ex/ Recall that $\vec{F}(\vec{r}) = -\frac{\vec{r}}{||\vec{r}||^2} = \nabla f$, where $f(\vec{r}) = \frac{\vec{r}}{||\vec{r}||}$. Let $C$ be the line segment connecting $A = (0,3,0)$ to $B = (4,3,0)$. We have

$$\int_C \vec{F} \cdot d\mathbf{r} = f(4,3,0) - f(0,3,0) = \frac{4}{5} - \frac{0}{3} = \frac{4}{15}.$$ 

Now recall that FTC I said that $\frac{d}{dx} \int_{x_0}^{x} F(t) \, dt = F(x)$. Here we want to say $\frac{d}{dx} \int_{x_0}^{x} \vec{F} \cdot d\mathbf{r} = \vec{F}(x)$, but without an assumption, $\int_{x_0}^{x} \vec{F} \cdot d\mathbf{r}$ makes no sense!
**FTC II**: Given \( f: S \rightarrow \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( S \), and \( C \subseteq S \) piecewise smooth oriented curve with \( [C] = \vec{B} - \vec{A} \),

\[
\int_C \nabla f \cdot \, d\vec{s} = f(\vec{B}) - f(\vec{A})
\]

---

**Example**

Recall that \( \vec{F}(\vec{x}) = -\frac{\vec{x}}{||\vec{x}||^2} = \nabla f \), where \( f(\vec{x}) = \frac{c}{||\vec{x}||} \). Let \( C \) be the line segment connecting \( \vec{A} = (0, 3, 0) \) to \( \vec{B} = (4, 3, 0) \).

We compute

\[
\int_C \vec{F} \cdot \, d\vec{s} = f(4, 3, 0) - f(0, 3, 0) = \frac{c}{4} - \frac{c}{3} = \frac{-c}{12}.
\]

Now recall that FTC I said that

\[
\frac{d}{dx} \int_{x_0}^x f(t) \, dt = F'(x).
\]

Here we want to say \( \nabla \int_{x_0}^x \vec{F} \cdot \, d\vec{s} = \vec{F}(x) \), but without an assumption, \( \int_{x_0}^x \vec{F} \cdot \, d\vec{s} \) makes no sense!
FTC II: Given \( f: \mathbb{R} \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathbb{R} \), and \( C = \overline{AB} \) piecewise smooth oriented curve with \( \delta C = \overrightarrow{B} - \overrightarrow{A} \),

\[
\int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)
\]

Ex/ Recall that \( \mathbf{F}(x) = -\frac{\mathbf{x}}{\|\mathbf{x}\|^2} = \nabla f \),
where \( f(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \). Let \( C \) be the line segment connecting \( A = (0, 3, 0) \) to \( B = (4, 3, 0) \).
We compute

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 3, 0) - f(0, 3, 0) = \frac{4}{5} - \frac{0}{5} = \frac{4}{5}
\]

Now recall that FTC I said that

\[
\frac{d}{dx} \int_{x_0}^x F(t) \, dt = F(x).
\]
Here we want to say \( \nabla \left( \int_{x_0}^x \frac{d}{dx} F(t) \, dt \right) = \frac{d}{dx} F(x) \), but without an assumption, \( \int_{x_0}^x \frac{d}{dx} F(t) \, dt \) makes no sense!

Let \( \mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n \) be continuous. We shall say that "\( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path on \( \mathbb{R}^n \)" if for each \( \overrightarrow{A}, \overrightarrow{B} \in \mathbb{R}^n \), \( \int_C \mathbf{F} \cdot d\mathbf{r} \) gives the same number regardless of what (piecewise smooth) \( C \) with \( \delta C = \overrightarrow{B} - \overrightarrow{A} \) is chosen.
**FTC II:** Given \( f : \mathbb{R} \to \mathbb{R} \) differentiable, with \( \frac{\partial f}{\partial x} \) continuous on \( S \), and \( C \subseteq S \) piecewise smooth oriented curve with \( [C] = \vec{B} - \vec{A} \),

\[
\int_C \frac{\partial f}{\partial x} \cdot dx = f(\vec{B}) - f(\vec{A})
\]

**Example:** Recall that \( \vec{F}(\vec{x}) = -\frac{\vec{x}}{||\vec{x}||^3} = \frac{\partial f}{\partial x} \), where \( f(\vec{x}) = \frac{\vec{x}}{||\vec{x}||} \). Let \( C \) be the line segment connecting \( \vec{A} = (0,3,0) \) to \( \vec{B} = (4,3,0) \).

We compute

\[
\int_C \vec{F} \cdot \, dx = f(4,3,0) - f(0,3,0) = \frac{4}{5} - \frac{3}{5} = -\frac{2}{5}
\]

Now recall that FTC I said that

\[
\frac{d}{dx} \int_{x_0}^{x} F(t) \, dt = F(x)
\]

here we want to say \( \frac{\partial}{\partial x} \int_{x_0}^{x} \vec{F} \cdot \, dx = \vec{F}(x) \), but without an assumption, \( \int_{x_0}^{x} \vec{F} \cdot \, dx \) makes no sense!

Let \( \vec{F} : S \to \mathbb{R}^n \) be continuous. We shall say that \( \int_C \vec{F} \cdot \, dx \) is independent of path on \( S \) if for each \( \vec{A}, \vec{B} \in S \),

\[
\int_C \vec{F} \cdot \, dx
\]

gives the same number regardless of what (piecewise smooth) \( C \) with \( [C] = \vec{B} - \vec{A} \) is chosen. Equivalently, we shall say that the vector field \( \vec{F} \) on \( S \) is conservative.
FTC II: Given $f: \mathbb{R}^n \to \mathbb{R}$ differentiable, with $\nabla f$ continuous on $\mathcal{B}$, and $C = \mathcal{B}$ piecewise smooth oriented curve with $\delta[C] = \hat{B} - \hat{A}$,

\[
\int_C \nabla f \cdot \, d\mathbf{r} = f(\hat{B}) - f(\hat{A}).
\]

Let $\vec{F}: \mathcal{B} \to \mathbb{R}^n$ be continuous. We shall say that "$\int_C \vec{F} \cdot d\mathbf{r}$ is independent of path on $\mathcal{B}$" if for each $\hat{A}, \hat{B} \in \mathcal{B}$, $\int_C \vec{F} \cdot d\mathbf{r}$ gives the same number regardless of what (piecewise smooth) $C$ with $\delta[C] = \hat{B} - \hat{A}$ is chosen. Equivalently, we shall say that the vector field $\vec{F}$ on $\mathcal{B}$ is conservative. So $\int_A^B \vec{F} \cdot d\mathbf{r}$ makes sense for every $\hat{A}, \hat{B} \in \mathcal{B}$ if and only if $\vec{F}$ is conservative (on $\mathcal{B}$).
**FTC II:** Given $f : \mathbb{R} \to \mathbb{R}$ differentiable, with $\nabla f$ continuous on $\mathbb{R}$, and $C = \mathbb{R}$ piecewise smooth oriented curve with $[C] = \vec{B} - \vec{A}$,

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{B}) - f(\vec{A}).$$

**FTC I:** When $\vec{F} : \mathbb{R} \to \mathbb{R}^n$ conservative and $\vec{A} \in \mathbb{R}$, $\varphi(\vec{x}) := \int_{\vec{A}}^{\vec{x}} \vec{F} \cdot d\vec{r}$ is well-defined and $\nabla \varphi = \vec{F}$.

Let $\vec{F} : \mathbb{R} \to \mathbb{R}^n$ be continuous. We shall say that $\int_C \vec{F} \cdot d\vec{r}$ is independent of path on $\mathbb{R}$ if for each $\vec{A}, \vec{B} \in \mathbb{R}$, $\int_C \vec{F} \cdot d\vec{r}$ gives the same number regardless of what (piecewise smooth) $C$ with $[C] = \vec{B} - \vec{A}$ is chosen. Equivalently, we shall say that the vector field $\vec{F}$ on $\mathbb{R}$ is conservative. So $\int_A^B \vec{F} \cdot d\vec{r}$ makes sense for every $\vec{A}, \vec{B} \in \mathbb{R}$ if and only if $\vec{F}$ is conservative (on $\mathbb{R}$).
FTC II: Given \( f : \mathbb{R} \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( A \), and \( C = \mathbb{R} \) piecewise smooth oriented curve with \( \mathcal{C} = \mathbb{R} - A \),

\[
\int_C \nabla f \cdot dr = f(\mathcal{B}) - f(\mathcal{A})
\]

FTC I: When \( \mathbb{F} : \mathbb{R} \to \mathbb{R}^n \) conservative and \( \mathcal{A} \in \mathbb{R} \), \( q(x) = \int_\mathcal{A}^x \mathbb{F} \cdot dr \) is well-defined and \( \nabla q = \mathbb{F} \).

Proof: Choose \( x \in \mathcal{A} \), \( r > 0 \) s.t. \( B(x;r) \subset \mathcal{A} \), and \( \hat{u} \) a unit vector. For \( 0 \leq h < r \),

\[
q(x + h\hat{u}) - q(x) = \int_x^{x+h\hat{u}} \mathbb{F} \cdot dr - \int_x^x \mathbb{F} \cdot dr
\]
FTC II: Given \( f : \mathbb{R} \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathbb{R} \), and \( C = \mathbb{R} \) piecewise smooth oriented curve with \( [C] = \vec{B} - \vec{A} \),

\[
\int_{C} \nabla f \cdot d\mathbf{r} = f(\vec{B}) - f(\vec{A})
\]

FTC I: When \( \vec{F} : \mathbb{R} \to \mathbb{R}^n \) conservative and \( \vec{A} \in \mathbb{R} \), \( \varphi(x) := \int_{\vec{A}}^{x} \vec{F} \cdot d\mathbf{r} \) is well-defined and \( \nabla \varphi = \vec{F} \).

Proof: Choose \( \vec{x} \in \mathbb{R} \), \( r > 0 \) s.t. \( B(\vec{x}, r) \subset \mathbb{R} \), and \( \hat{u} \) a unit vector: for \( 0 \leq h < r \),

\[
\varphi(\vec{x} + h\hat{u}) - \varphi(\vec{x}) = \int_{\vec{A}}^{\vec{x} + h\hat{u}} \vec{F} \cdot d\mathbf{r} - \int_{\vec{A}}^{\vec{x}} \vec{F} \cdot d\mathbf{r}
\]

\[
= \int_{0}^{h} \nabla \varphi(\vec{x} + t\hat{u}) \cdot \hat{u} dt
\]

where \( \frac{d}{dt} \varphi(t) = \vec{F}(\vec{x} + t\hat{u}) \cdot \hat{u} dt \).
FTC II: Given \( f: \mathbb{R} \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathcal{S} \), and \( C = \mathcal{S} \) a piecewise smooth oriented curve with \( \partial [C] = \mathcal{B} - \mathcal{A} \),

\[
\int_C \nabla f \cdot d\mathbf{x} = f(\mathcal{B}) - f(\mathcal{A})
\]

FTC I: When \( F: \mathbb{R} \to \mathbb{R}^n \) is conservative and \( \mathcal{A}, \mathcal{B} \in \mathcal{S} \), \( \varphi(x) := \int_{\mathcal{A}}^{x} F \cdot d\mathbf{r} \) is well-defined and \( \nabla \varphi = F \).

Proof: Choose \( \mathbf{x} \in \mathcal{S} \), \( r > 0 \) s.t. \( B(x; r) \subset \mathcal{S} \), and \( \mathbf{u} \) a unit vector: for \( 0 \leq h < r \),

\[
\varphi(x + h \mathbf{u}) - \varphi(x) = \int_{x}^{x + h \mathbf{u}} \nabla \varphi \cdot d\mathbf{r} = \int_{x}^{x + h \mathbf{u}} F \cdot d\mathbf{r}
\]

\[
= \int_{0}^{h} \int_{x}^{x + tu} F \cdot d\mathbf{r} \cdot dt = \int_{0}^{h} F(x + tu) \cdot \mathbf{u} \cdot dt
\]
**FTC II:** Given $f : \mathbb{R} \to \mathbb{R}$ differentiable, with $\frac{\partial f}{\partial x}$ continuous on $\mathbb{R}$, and $C = \bar{A}B$ a piecewise smooth oriented curve with $\int_C \nabla f \cdot \mathbf{r} = f(\bar{B}) - f(\bar{A})$, then
\[ \int_C \frac{\partial f}{\partial x} \, dr = f(\bar{B}) - f(\bar{A}). \]

**FTC I:** Given $F : \mathbb{R} \to \mathbb{R}^n$ conservative and $\bar{A} \in \mathbb{R}^n$, $\varphi(x) := \int_{\bar{A}}^x F \cdot dr$ is well-defined and $\nabla \varphi = F$.

**Proof:** Choose $\hat{x} \in \mathbb{R}$, $r > 0$ s.t. $B(\hat{x},r) \subseteq \mathbb{R}$, and $\hat{u}$ a unit vector; for $0 \leq h < r$,
\[ \varphi(\hat{x} + h\hat{u}) - \varphi(\hat{x}) = \int_{\hat{x}}^{\hat{x} + h\hat{u}} F \cdot dr = \int_{\hat{x}}^{\hat{x} + h\hat{u}} F \cdot \hat{u} \, dt = \int_{0}^{h} F(\hat{x} + t\hat{u}) \cdot \hat{u} \, dt. \]
**FTC II**: Given \( f: \mathbb{R} \rightarrow \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathbb{R} \), and \( C = \gamma \) piecewise smooth oriented curve with \( \gamma[C] = \vec{B} - \vec{A} \),

\[
\int_C \nabla f \cdot d\mathbf{r} = f(\vec{B}) - f(\vec{A})
\]

**FTC I**: When \( \vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n \) conservative and \( \vec{A} \in \mathbb{R}^n \), \( \varphi(x) := \int_{\vec{A}}^{\vec{x}} \vec{F} \cdot d\mathbf{r} \) is well-defined and \( \nabla \varphi = \vec{F} \).

**Proof**: Choose \( \vec{x}_0 \in \mathbb{R}^n \), \( r > 0 \) s.t. \( B(\vec{x}_0;r) \subset \mathbb{R}^n \), and \( \hat{u} \) a unit vector: for \( 0 \leq h < r \),

\[
\varphi(\vec{x} + h\hat{u}) - \varphi(\vec{x}) = \int_{\vec{x}}^{\vec{x} + h\hat{u}} \vec{F} \cdot d\mathbf{r} = \int_{\vec{x}}^{\vec{x} + h\hat{u}} \vec{F} \cdot \hat{u} \, dt
\]

\[
= \int_{0}^{h} \vec{F}(\vec{x} + t\hat{u}) \cdot \hat{u} \, dt = \int_{0}^{h} \vec{F}(\vec{x} + t\hat{u}) \cdot \hat{u} \, dt
\]

\[
\Rightarrow \quad \varphi(\vec{x} + h\hat{u}) - \varphi(\vec{x}) = \int_{0}^{h} \vec{F}(\vec{x} + t\hat{u}) \cdot \hat{u} \, dt
\]

\[
\nabla \varphi = \vec{F}
\]

**1-var FTC I**

\[
\frac{d\varphi}{dx}(\vec{x}) = \lim_{h \to 0} \frac{\varphi(\vec{x} + h\hat{u}) - \varphi(\vec{x})}{h}
\]

\[
= \lim_{h \to 0} \frac{\varphi(\vec{x} + h\hat{u}) - \varphi(\vec{x})}{h}
\]

\[
= \varphi'(\vec{x}) = f_k(\vec{x})
\]
**FTC II**: Given \( f : \mathbb{R} \to \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathbb{R} \), and \( C = \overline{A'B} \) piecewise smooth oriented curve with \( \delta[C] = \overline{B'-A} \),

\[
\int_{C} \nabla f \cdot \mathbf{dr} = f(B) - f(A')
\]

**FTC I**: When \( \nabla F : \mathbb{R} \to \mathbb{R}^n \) conservative and \( \overline{A'B} \subseteq \mathbb{R} \), \( \varphi(x) := \int_{\overline{A'B}} F \cdot d\mathbf{r} \) is well-defined and \( \nabla \varphi = F \).

**Proof**: Choose \( \hat{x} \in \mathbb{B} \), \( r > 0 \) s.t. \( B(\hat{x}; r) \subseteq \mathbb{B} \), and \( \mathbf{u} \) a unit vector: for \( 0 \leq h < r \),

\[
\varphi(x + h\mathbf{u}) - \varphi(x) = \int_{x}^{x + h\mathbf{u}} \nabla \varphi \cdot \mathbf{dr} = \int_{x}^{x + h\mathbf{u}} F \cdot d\mathbf{r} = \int_{0}^{h} F(x + t\mathbf{u}) \cdot \mathbf{u} dt
\]

\[
\dot{\varphi}(x) = \frac{d}{dt} \varphi(x + t\mathbf{u}) = \sum_{i=1}^{n} F(x + t\mathbf{u})_{i} \frac{d}{dt} u_{i} = \sum_{i=1}^{n} F(x + t\mathbf{u})_{i} = \nabla \varphi(x) \cdot \mathbf{u}
\]
FTC II: Given \( f: \mathbb{R} \rightarrow \mathbb{R} \) differentiable, with \( \nabla f \) continuous on \( \mathbb{R} \), and \( C = \beta \mathbf{A} \) piecewise smooth oriented curve with \( \nabla \mathbb{C} = \mathbf{B} - \mathbf{A} \),

\[
\int_C \frac{\nabla f}{\kappa} \cdot d\mathbf{r} = f(\mathbf{B}) - f(\mathbf{A})
\]

FTC I: When \( \nabla \mathbb{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) conservative and \( \mathbf{A} \in \mathbb{R}^3 \), \( \varphi(\mathbf{x}) := \int_{\mathbf{A}}^{\mathbf{x}} \mathbb{F} \cdot d\mathbf{x} \) is well-defined and \( \nabla \varphi = \mathbb{F} \).

Proof: Choose \( \mathbf{x} \in \mathbb{R}, r > 0 \) s.t. \( B(\mathbf{x}, r) \subset \mathbb{R} \), and \( \mathbf{u} \) a unit vector: for \( 0 \leq h < r \),

\[
\varphi(\mathbf{x} + h\mathbf{u}) - \varphi(\mathbf{x}) = \int_{\mathbf{A}}^{\mathbf{x} + h\mathbf{u}} \mathbb{F} \cdot d\mathbf{x} - \int_{\mathbf{A}}^{\mathbf{x}} \mathbb{F} \cdot d\mathbf{x}
\]

\[
= \int_{\mathbf{x}}^{\mathbf{x} + h\mathbf{u}} \mathbb{F} \cdot d\mathbf{x} = \int_0^h \mathbb{F}(\mathbf{x} + t\mathbf{u}) \cdot \mathbf{u} dt
\]

\[
\mathbf{u} = \mathbf{e}_k
\]

\[
\Rightarrow \quad \varphi(\mathbf{x} + h\mathbf{e}_k) - \varphi(\mathbf{x}) = \int_0^h \mathbb{F}(\mathbf{x} + t\mathbf{e}_k) \cdot \mathbf{e}_k dt
\]

\[
F = (f_1, ..., f_n) = \int_0^h f_k(\mathbf{x} + t\mathbf{e}_k) dt
\]

Call this \( g(h) \)

\[
\Rightarrow \quad g'(h) = f_k(\mathbf{x} + h\mathbf{e}_k)
\]

1-var FTC I

\[
\frac{\partial \varphi}{\partial x_k}(\mathbf{x}) = \lim_{h \rightarrow 0} \frac{\varphi(\mathbf{x} + h\mathbf{e}_k) - \varphi(\mathbf{x})}{h}
\]

\[
= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = g'(0) \quad (\text{constant})
\]

\[
= g'(0) = f_k(\mathbf{x})
\]

\[
\Rightarrow \quad \nabla \varphi = \mathbb{F}
\]

So FTC I says conservative fields have a potential function. We'll talk about how to construct this potential function computationally next time.
A simple application to DES:
A simple application to DEs:

A differential of the form

\[ P(x,y)\, dx + Q(x,y)\, dy \]

is called exact if \[ P = \frac{\partial g}{\partial x} \] and

\[ Q = \frac{\partial g}{\partial y} \]

for some function \( g(x,y) \).
A simple application to DEs:

A differential of the form

\[ P(x,y) \; dx + Q(x,y) \; dy \quad (\ast) \]

is called exact if \( P = \frac{\partial g}{\partial x} \) and \( Q = \frac{\partial g}{\partial y} \) for some function \( g(x,y) \).

Writing \((\ast) = 0\) gives the DE

\[ P + Q \; \frac{dy}{dx} = 0. \quad (\ast\ast) \]
A simple application to DEs:

A differential of the form

\[ P(x,y) \, dx + Q(x,y) \, dy \] (*)

is called exact if \( P = \frac{\partial g}{\partial x} \) and \( Q = \frac{\partial g}{\partial y} \) for some function \( g(x,y) \).

Writing \((*) = 0\) gives the DE

\[ P + Q \frac{dy}{dx} = 0. \] (**)

**Proposition**: Assume \((*)\) is exact. Then the solutions

\[ y = Y(x) \] of (**)

are precisely the functions whose graphs are level curves of \( g \).
A simple application to DEs:

A differential of the form

\[ P(x,y) \, dx + Q(x,y) \, dy \]  \((*)\)

is called exact if \( P = \frac{\partial g}{\partial x} \) and \( Q = \frac{\partial g}{\partial y} \) for some function \( g(x,y) \).

Writing \((*) = 0\) gives the DE

\[ P + Q \frac{dy}{dx} = 0. \] \((***)\)

**Proposition:** Assume \((***)\) is exact. Then the solutions

\[ y = Y(x) \] of \((***)\)

are precisely the functions whose graphs are level curves of \( g \).

**Proof:** Writing \( g(x) = g(x, Y(x)) \), we have

\[
g'(x) = \frac{\partial g}{\partial x} (x, Y(x)) + \frac{\partial g}{\partial y} (x, Y(x)) \cdot Y'(x)
\]

\[
= P(x, Y(x)) + Q(x, Y(x)) \frac{dy}{dx}
\]

and so \( Y(x) \) solves \((***) \iff g \) is constant \( \iff (x, Y(x)) \) traces a level curve. \(\Box\)
A simple application to DEs:

A differential of the form

\[ P(x, y) \, dx + Q(x, y) \, dy \] (*1)

is called exact if \( P = \frac{\partial g}{\partial x} \) and \( Q = \frac{\partial g}{\partial y} \) for some function \( g(x, y) \).

Writing \((*) = 0\) gives the DE

\[ P + Q \, \frac{dy}{dx} = 0. \] (***)

**Proposition:** Assume \((*)\) is exact. Then the solutions

\[ y = Y(x) \] of \((***)\)

are precisely the functions whose graphs are level curves of \( g \).

**Proof:** Writing \( g(x) = f(x, Y(x)) \), we have

\[ g'(x) = \frac{\partial f}{\partial x} (x, Y(x)) + \frac{\partial f}{\partial y} (x, Y(x)) \frac{dy}{dx} \]

\[ = P(x, Y(x)) + Q(x, Y(x)) \frac{dy}{dx} \]

and so

\[ Y(x) \text{ solves } (***) \iff g \text{ is constant} \iff (x, Y(x)) \text{ traces a level curve.} \]

**Ex:** We can use this to solve equations of the form \( \frac{dy}{dx} = G(x, y) \). For example,

\[ \frac{dy}{dx} = -\frac{y}{2x} \]
A simple application to DEs:

A differential of the form

\[ P(x,y) \, dx + Q(x,y) \, dy \quad (*) \]

is called exact if \( P = \frac{\partial g}{\partial x} \) and \( Q = \frac{\partial g}{\partial y} \) for some function \( g(x,y) \).

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\[ P + Q \frac{dy}{dx} = 0. \quad (**) \]

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\[ y = Y(x) \quad \text{of (**)} \]

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Proof: Writing \( g(x) = g(x, Y(x)) \), we have

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\[ = P(x, Y(x)) + Q(x, Y(x)) \frac{dy}{dx} \quad \text{and so} \]

\( Y(x) \) solves \((**) \) \( \iff \) \( g \) is constant

\[ \iff (x, Y(x)) \text{ traces a level curve.} \]

Ex: We can use this to solve equations of the form \( \frac{dy}{dx} = G(x,y) \). For example,

\[ \frac{dy}{dx} = -\frac{y}{2x} \quad \implies \quad y \, dx + 2x \, dy = 0 \]

not exact \(-\text{so doesn't work?}\)
A simple application to DEs:

A differential of the form

\[ P(x, y) \, dx + Q(x, y) \, dy \quad (\ast) \]

is called exact if \( P = \frac{\partial g}{\partial x} \) and \( Q = \frac{\partial g}{\partial y} \) for some function \( g(x, y) \).

Writing \((\ast) = 0\) gives the DE

\[ P + Q \frac{dy}{dx} = 0. \quad (\ast\ast) \]

**Proposition:** Assume \((\ast)\) is exact. Then the solutions

\[ y = \psi(x) \quad \text{of} \quad (\ast\ast) \]

are precisely the functions whose graphs are level curves of \( g \).

**Proof:** Writing \( g(x) = g(x, y(x)) \), we have

\[
g'(x) = \frac{\partial g}{\partial x}(x, y(x)) + \frac{\partial g}{\partial y}(x, y(x)) \frac{dy}{dx}.
\]

Thus

\[ P(x, y(x)) + Q(x, y(x)) \frac{dy}{dx} \quad \text{and so} \]

\( y(x) \) solves \((\ast) \iff g \) is constant

\[ \iff (x, y(x)) \text{ traces a level curve.} \]

**Example:** We can use this to solve equations of the form \( \frac{dy}{dx} = g(x, y) \). For example,

\[ \frac{dy}{dx} = -\frac{y}{2x} \quad \rightarrow \quad y \, dx + 2x \, dy = 0 \]

**not exact** — so doesn't work?\n
**not so fast!**

Multiply by \( y \):

\[
y^2 \, dx + 2xy \, dy = 0
\]

\[
\frac{2x^2}{3} y^2 + \frac{2}{3} xy^2 = \text{ (so } y = x^{\frac{3}{2}})\]
A simple application to DEs:

A differential of the form

\[ P(x,y) \, dx + Q(x,y) \, dy \]  

is called exact if \( P = \frac{\partial g}{\partial x} \) and \( Q = \frac{\partial g}{\partial y} \) for some function \( g(x,y) \).

Writing \((*) = 0 \) gives the DE

\[ P + Q \frac{dy}{dx} = 0. \]  

\((***)\)

Proof: Writing \( g(x) = g(x, Y(x)) \), we have

\[
g'(x) = \frac{\partial g}{\partial x} (x, Y(x)) + \frac{\partial g}{\partial y} (x, Y(x)) \cdot Y'(x)
= P(x, Y(x)) + Q(x, Y(x)) \frac{dy}{dx}
\]

and so

\( Y(x) \) solves \((***) \) \( \iff \) \( g \) is constant

\( \iff \) \((x, Y(x))\) traces a level curve.

Example: We can use this to solve equations of the form \( \frac{dy}{dx} = G(x,y) \). For example,

\[
\frac{dy}{dx} = -\frac{y}{2x} \quad \Rightarrow \quad y \, dx + 2x \, dy = 0
\]

not exact - so doesn't work!

not so fast!

multiply by \( y \)

\[
y^2 \, dx + 2xy \, dy = 0
\]

\[\sqrt{\frac{2x^2}{y^4} \cdot \frac{3y^2}{4}} \quad (so \ y^2 = xy^2)\]

level sets are \( xy^2 = C \) \( \Rightarrow \)

solutions are \( Y(x) = \sqrt{\frac{C}{x}} \).