Lecture 36: Independence of path

Cast of characters

VECTOR FIELD A

The wind velocity field around my horse in St. Louis is the spring:

\[ \vec{F}_A(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right) = \frac{-y \hat{e}_i + x \hat{e}_j}{r^2} \]

(Clearly, “there’s no place like home.”)

\[ \oint_C \vec{F}_A \cdot d\vec{s} = \int_0^{2\pi} \left( -\sin t, \cos t \right) \cdot \left( -\sin t, \cos t \right) dt = \int_0^{2\pi} 1 \, dt = 2\pi \]

VECTOR FIELD B

Gravitational attraction in Flatland:

\[ \vec{F}_B(x, y) = \left( \frac{-x}{(x^2 + y^2)^{3/2}}, \frac{-y}{(x^2 + y^2)^{3/2}} \right) = \frac{-x \hat{e}_i - y \hat{e}_j}{r^3} = \nabla \varphi, \]

\[ \varphi(x, y) = \frac{1}{r} \quad (\text{by FTC I, } \oint_C \vec{F}_B \cdot d\vec{s} = 0) \]

VECTOR FIELD C

\[ \vec{F}_C(x, y) = \left( x^2, x^2 - y^2 \right) \]

- defined on \( \mathbb{R}^2 \)
- not conservative (example in Lect. 34)
As we did yesterday, we assume

- $\mathcal{A} \subseteq \mathbb{R}^n$ is a connected open set
- $\vec{F}: \mathcal{A} \rightarrow \mathbb{R}^n$ is continuous

**Theorem 1:** The following are equivalent:

(a) $\vec{F}$ is a gradient field on $\mathcal{A}$;
(b) $\vec{F}$ is conservative on $\mathcal{A}$ (i.e. $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ is independent of path)
(c) $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ for all closed $\mathcal{C} \subseteq \mathcal{A}$.

**Proof:**

(b) $\Rightarrow$ (a): FTC I [think $\nabla \left( \int_{x_0}^{x} \vec{F} \cdot d\vec{x} \right) = \vec{F}$]

(a) $\Rightarrow$ (c): (Corollary to) FTC II $\left[ \int_{\mathcal{A}} \nabla \phi \cdot d\vec{r} = \phi(B) - \phi(A) = 0 \text{ if } \mathcal{A} = \emptyset \right]$

(c) $\Rightarrow$ (b): if $C_1, C_2$ are both paths from $A \rightarrow B$,

then $-C_2$ is a path from $B \rightarrow A$ and $C_1 - C_2$ is closed:

So $0 = \int_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} \Rightarrow$ independence of path.

**Theorem 2:** If $\vec{F} = (f_1, \ldots, f_n)$ is $C^2$, and $\vec{F} = \nabla \phi$ on $\mathcal{A}$, then

\[ \frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k} \quad \forall j, k \text{ (on all of } \mathcal{A}). \]

**Proof:** Use Clairaut's Thm.: Since $f_k = \frac{\partial \phi}{\partial x_k}$,

\[ \frac{\partial f_k}{\partial x_j} = \frac{\partial}{\partial x_j} \left( \frac{\partial \phi}{\partial x_k} \right) = \frac{\partial^2 \phi}{\partial x_j \partial x_k} = \frac{\partial^2 \phi}{\partial x_k \partial x_j}, \]

This gives an obstruction to conservativity: if (2) fails, then $\vec{F}$ isn't a gradient field so can't be conservative.
Notice that in both results, you have to pay attention to the domain $\mathcal{D}$. It is entirely possible that the equivalent conditions in Theorem 1 FAIL on $\mathcal{D}$ but HOLD on a subset. On the other hand, with Theorem 2, if $(\star)$ fails on $\mathcal{D}$, it will typically fail on all open subsets. In other words, if $\frac{\partial f}{\partial y} \neq \frac{\partial f}{\partial x}$ for some $j, k$, then $F^j$ is not even locally a gradient field.

**Our Examples**

\[ F_A = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \]

\[
\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{2}{x^2}\left(\frac{x}{x^2+y^2}\right) \Rightarrow (\star) \text{ holds } \Rightarrow \text{ "inconclusive" (but see below)}
\]

\[ F_B = \left(\frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}}\right) \]

\[
\frac{\partial}{\partial y}\left(\frac{-x}{(x^2+y^2)^{3/2}}\right) = \frac{3xy}{(x^2+y^2)^{5/2}} = \frac{1}{x^2}\left(\frac{-y}{(x^2+y^2)^{3/2}}\right) \Rightarrow (\star) \text{ holds } \Rightarrow \text{ inconclusive.}
\]

But, we already knew $\overrightarrow{F_B}$ is conservative, so lemma (2) would hold.

\[ F_C = (xy^2, x^2-y^2) \]

\[
\frac{\partial}{\partial y} xy^2 = 2xy \neq 2x = \frac{\partial}{\partial x} (x^2-y^2) \Rightarrow (\star) \text{ fails } \Rightarrow \overrightarrow{F_C} \text{ not a gradient field (even on small circles)}
\]

We know $\overrightarrow{F_A}$ isn't a gradient field on $\mathcal{D} = (\mathbb{R}^2 \setminus \mathbb{S}^2)$, b/c of the nonzero loop integral. But perhaps the equality of partials tells us something?
To find out, let's say we are given $F = (P, Q)$ on a rectangle $R \subset \mathbb{R}^2$, with $P, Q \in C^1$ satisfying
\[
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.
\]

Set $h(x, y) := \int_0^x P(u, y) \, du \in C^1(R)$

Then $\frac{\partial h}{\partial x} = P$, and

$F - \nabla h = (0, Q - \frac{\partial h}{\partial y})$

But $\frac{\partial}{\partial x} (Q - \frac{\partial h}{\partial y}) = \frac{\partial Q}{\partial x} - \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial Q}{\partial x} - \frac{\partial^2 h}{\partial y \partial x} - \frac{\partial h}{\partial y} = 0$ by assumption,

and so $Q - \frac{\partial h}{\partial y}$ is a function $g(y)$ of $y$ (because constant in the $x$-direction). Let $G(y)$ be an antiderivative of $g(y)$.

Then $\nabla (h + G) = (P, \frac{\partial h}{\partial y}) + (0, g) = (P, \frac{\partial h}{\partial y} + (Q - \frac{\partial h}{\partial y})) = (P, Q) = F$ !

Hence $F$ is conservative on $R$.

The argument extends to any connected open set which is also simply connected, i.e. has no holes.
The problem with a non-simply connected set is that the formula for \( h \) can develop a discontinuity on one side of a hole, so that our "\( h + \mathbf{G} \)" whose gradient was \( \bar{\mathbf{F}} \) on the hole-less rectangle, no longer works.

**Upshot for Example A:** On any subset of \( S = \mathbb{R}^2 \setminus \{0\} \) which is simply connected, \( \int_A \mathbf{F} \) is conservative: on the right half-plane, ... or even on the slit plane:

If we define \( \theta(x,y) \) as shown, with range \((-\pi, \pi)\), then \( y/x = \tan \theta \Rightarrow \)

\[
\frac{1}{x} = \frac{\partial}{\partial y} \frac{y}{x} = \sec^2 \theta \quad \Rightarrow \quad \frac{\partial \theta}{\partial y} = \cos^2 \theta = \frac{x^2}{x^2 + y^2} = \frac{x}{r}.
\]

\[
-\frac{y}{x^2} = \frac{\partial}{\partial x} \frac{y}{x} = \sec^2 \theta \frac{\partial \theta}{\partial x} \quad \Rightarrow \quad \frac{\partial \theta}{\partial x} = -\frac{y}{x^2} = -\frac{y}{r^2} = \frac{-y}{r^2}.
\]

\( \Rightarrow \frac{\partial \theta}{\partial \mathbf{x}} = \left(-\frac{y}{r^2}, \frac{x}{r^2}\right) = \bar{\mathbf{F}}_A. \)
Ex./ Determine whether \( \vec{F} = (P, Q) \) with
\[
\begin{align*}
P &= 4x^3 + 9x^2y^2 \\
Q &= 6x^3y + 6y^5
\end{align*}
\]
is conservative.

If so, find the function \( \varphi \) with \( \nabla \varphi = \vec{F} \).

First, \( P_y = 18x^2y = Q_x \), and \( \vec{F} \) is defined on all of \( \mathbb{R}^2 \), which has no holes. Thus, \( \vec{F} \) is conservative.

To find \( \varphi \), write \( P = p_x \) and \( Q = p_y \). Antidifferentiating the first with respect to \( x \) gives
\[
x^4 + 3x^3y^2 + C_1(y) = p(x, y),
\]
for SOME function \( C_1 \) of \( y \) alone.

Antidifferentiating the second w.r.t. \( y \) gives
\[
3x^3y^2 + y^6 + C_2(x) = p(x, y);
\]
so \( C_1(y) = y^6 \) (and \( C_2(x) = x^4 \)).

Ex./ Suppose you know that \( \vec{F} = (P, Q, R) \) with
\[
\begin{align*}
P &= e^x \cos y + yz \\
Q &= x^2 - e^x \sin y \\
R &= xy + 2z
\end{align*}
\]
is conservative.

Compute \( \int_C \vec{F} \cdot d\vec{r} \), where \( C \) is some path from \( \vec{A} = (0, \pi, 2) \) to \( \vec{B} = (1, \frac{\pi}{2}, -3) \).
Conservativity is plausible since \( P_y = Q_x \), \( P_x = R_y \), and \( Q_x = R_y \), while \( R^3 \) is simply connected (though we haven't yet stated a result like this).

Now antidifferentiate:

- \( P_x = P \) \( \implies \) \( p = e^x \cos y + xy + C_1(y,t) \)
- \( P_y = Q \) \( \implies \) \( p = e^x \sin y + xy + C_2(x,t) \)
- \( P_z = R \) \( \implies \) \( p = xy + 2z + C_3(x,y) \)

\( \implies \) \( p = e^x \cos y + xy + z + e^2 \) (plus a constant)

\( \implies \) \( \int_C \vec{F} \cdot d\vec{s} = p(B) - p(A) = (9 - \frac{3\pi}{2}) - 3 = 6 - \frac{3\pi}{2} \).

We conclude with the \( n \)-dimensional converse to Theorem 2 (which we proved so far for when \( n = 2 \) and \( \mathcal{D} \) is an open rectangle).

**Definition:** \( \mathcal{D} \) is said to be convex if for every \( \vec{x}_0, \vec{x}_1 \in \mathcal{D} \), it contains the line segment with \( \vec{x}_0, \vec{x}_1 \) as endpoints. **proof next time**

**Theorem 3:** Suppose \( \mathcal{D} \) is open \& convex, \( \vec{F} = (f_1, \ldots, f_n) : \mathcal{D} \rightarrow \mathbb{R}^n \) is \( C^1 \), and \( \frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i} \) (\( i,j \)). Then \( \vec{F} \) is conservative.

Actually this is still true for \( \mathcal{D} \) simply connected (much more general than convex), but we won't prove that.