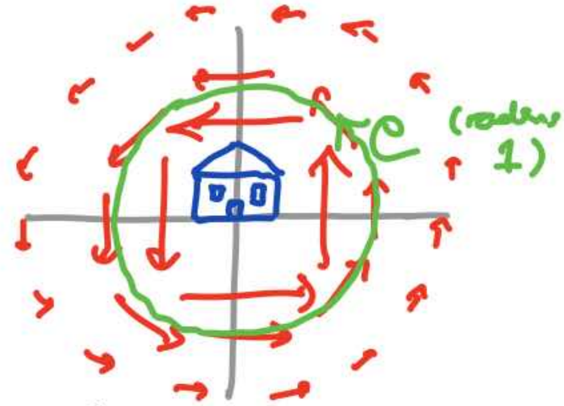
Lecture 36: Independence of path

Cast of characters

VECTOR FIELD A



The wind velocity field around my house in St. Louis in the spring:

$$\vec{F}_A(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) = \frac{-y\,\hat{e}_1 + x\,\hat{e}_2}{r^2}$$

(Clearly, "there's no place like home.")

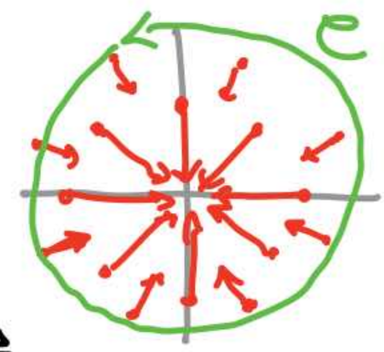
- $\vec{F}_A$ tangent to circles
- defined on $\mathbb{R}^2 \setminus \{\vec{0}\}$
- not conservative

e.g.

$$\int_{\mathcal{C}} \vec{F}_A \cdot d\vec{r} = \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt = \int_0^{2\pi} 1\,dt = 2\pi$$

$\vec{r}(t) = (\sin t, \cos t)$

VECTOR FIELD B



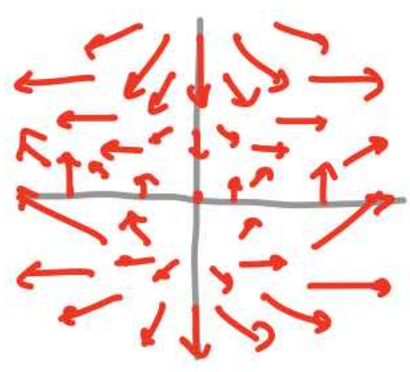
Gravitational attraction in Flatland:

$$\vec{F}_B(x,y) = \left(\frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}}\right) = \frac{-x\,\hat{e}_1 - y\,\hat{e}_2}{r^3} = \vec{\nabla}\varphi,$$

$\varphi(x,y) = \frac{1}{r}$ (by FTC II, $\int_{\mathcal{C}} \vec{F}_B \cdot d\vec{r} = 0$.)

- defined on $\mathbb{R}^2 \setminus \{\vec{0}\}$
- conservative

VECTOR FIELD C



$$\vec{F}_C(x,y) = (xy^2, x^2 - y^2)$$

- defined on $\mathbb{R}^2$
- not conservative (example in Lect. 34)

in fact, small changes in $\mathcal{C}$ (keeping endpts. fixed) change $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$

As we did yesterday, we assume

- $\mathcal{S} \subset \mathbb{R}^n$ is a connected open set
- $\vec{F} : \mathcal{S} \to \mathbb{R}^n$ is continuous

<u>Theorem 1</u>: The following are equivalent:

(a) $\vec{F}$ is a gradient field <u>on $\mathcal{S}$</u>;

(b) $\vec{F}$ is conservative <u>on $\mathcal{S}$</u> (i.e. $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$ is independent of path)

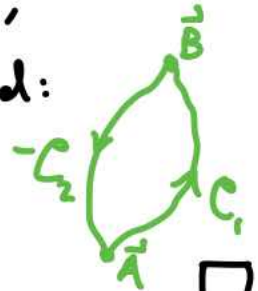
(c) $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ for all <u>closed</u> $\mathcal{C} \subset \mathcal{S}$.

<u>Proof</u>: (b) $\Rightarrow$ (a) is FTC I $\quad \left[\text{think } \vec{\nabla}\Big(\underbrace{\int_{\vec{x}_0}^{\vec{x}} \vec{F} \cdot d\vec{r}}_{\text{well-def'd since } \vec{F} \text{ conservative}}\Big) = \vec{F} \right]$

(a) $\Rightarrow$ (c) is (Corollary to) FTC II $\quad \left[\int_{\vec{A}}^{\vec{B}} \vec{\nabla}\varphi \cdot d\vec{r} = \varphi(\vec{B}) - \varphi(\vec{A}) = 0 \text{ if } \vec{A} = \vec{B} \right]$

(c) $\Rightarrow$ (b): if $\mathcal{C}_1, \mathcal{C}_2$ are both paths from $\vec{A}$ to $\vec{B}$,

then $-\mathcal{C}_2$ is a path from $\vec{B}$ to $\vec{A}$ and $\mathcal{C}_1 - \mathcal{C}_2$ is closed:



So $0 = \int_{\mathcal{C}_1 - \mathcal{C}_2} \vec{F} \cdot d\vec{r} = \int_{\mathcal{C}_1} \vec{F} \cdot d\vec{r} - \int_{\mathcal{C}_2} \vec{F} \cdot d\vec{r} \Rightarrow$ independence of path. $\square$

<u>Theorem 2</u>: If $\vec{F} = (f_1, \dots, f_n)$ is C^1, and $\vec{F} = \vec{\nabla}\varphi$ <u>on $\mathcal{S}$</u>, then

$$(*) \qquad \frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k} \quad \forall j, k \quad (\text{on all of } \mathcal{S}).$$

<u>Proof</u>: Use Clairaut's Thm.: since $f_k = \frac{\partial \varphi}{\partial x_k}$,

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial^2 \varphi}{\partial x_j \partial x_k} = \frac{\partial^2 \varphi}{\partial x_k \partial x_j} = \frac{\partial f_j}{\partial x_k}. \qquad \square$$

This gives an <u>obstruction</u> to conservativity: if $(*)$ fails, then $\vec{F}$ isn't a gradient field so can't be conservative.

Notice that in both results, you have to pay attention to the domain $\mathcal{S}$. It is entirely possible that the equivalent conditions in Theorem 1 FAIL on $\mathcal{S}$ but HOLD on a subset. On the other hand, with Theorem 2, if (*) fails on $\mathcal{S}$, it will typically fail on all open subsets. In other words, if $\partial f_k/\partial x_j \neq \partial f_j/\partial x_k$ for some j, k, then $\vec{F}$ is not even locally a gradient field.

Our Examples

$$\vec{F}_A = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) \Rightarrow (*) \text{ holds} \Rightarrow \text{"inconclusive"}$$

(but see below)

$$\vec{F}_B = \left(\frac{-x}{(x^2+y^2)^{3/2}}, \frac{-y}{(x^2+y^2)^{3/2}}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{-x}{(x^2+y^2)^{3/2}}\right) = \frac{3xy}{(x^2+y^2)^{5/2}} = \frac{\partial}{\partial x}\left(\frac{-y}{(x^2+y^2)^{5/2}}\right) \rightarrow (*) \text{ holds} \rightarrow \text{inconclusive.}$$

BUT: we already know $\vec{F}_B$ is conservative, so knew (*) would hold

$$\vec{F}_C = (xy^2, x^2-y^2)$$

$$\frac{\partial}{\partial y} xy^2 = 2xy \neq 2x = \frac{\partial}{\partial x}(x^2-y^2) \Rightarrow (*) \text{ fails} \xRightarrow{\text{Thm. 2}} \vec{F}_C \text{ not a gradient field}$$

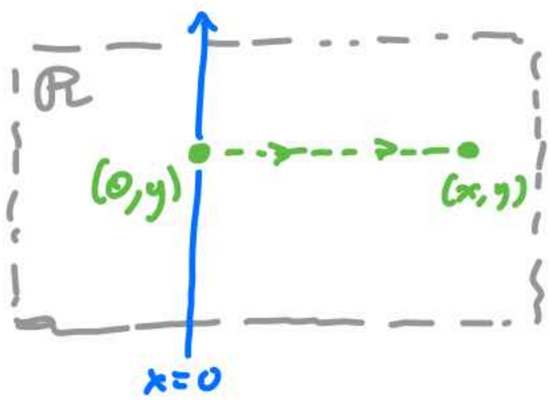
(even on small disks)

We know $\vec{F}_A$ isn't a gradient field on $\mathcal{S} = \mathbb{R}^2 \setminus \{\vec{0}\}$, b/c of the nonzero loop integral. But perhaps the equality of partials tells us something?

To find out, let's say we are given $\vec{F} = (P, Q)$ on a rectangle $\mathcal{R} \subset \mathbb{R}^2$, with P, Q C^1 satisfying

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Set $h(x,y) := \int_0^x P(u,y)\, du \in C^1(\mathcal{R})$



Then $\frac{\partial h}{\partial x} = P$, and

$$\underset{(P,Q)}{F} - \underset{\left(P, \frac{\partial h}{\partial y}\right)}{\vec{\nabla} h} = \left(0, Q - \frac{\partial h}{\partial y}\right)$$

But $\frac{\partial}{\partial x}\left(Q - \frac{\partial h}{\partial y}\right) = \frac{\partial Q}{\partial x} - \frac{\partial^2 h}{\partial x \partial y}$

$$\overset{\text{Clairaut}}{=} \frac{\partial Q}{\partial x} - \frac{\partial^2 h}{\partial y \partial x} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \text{ by assumption},$$

and so $Q - \frac{\partial h}{\partial y}$ is a function $g(y)$ of y (because constant in the x-direction). Let $G(y)$ be an antiderivative of $g(y)$.

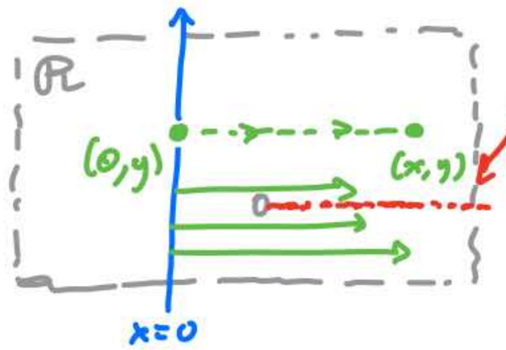
Then
$$\vec{\nabla}(h+G) = \left(P, \frac{\partial h}{\partial y}\right) + (0, g) = \left(P, \frac{\partial h}{\partial y} + \left(Q - \frac{\partial h}{\partial y}\right)\right)$$
$$= (P, Q) = \vec{F} \quad !!$$

Hence $\vec{F}$ is conservative on $\mathcal{R}$.

The argument extends to any connected open set

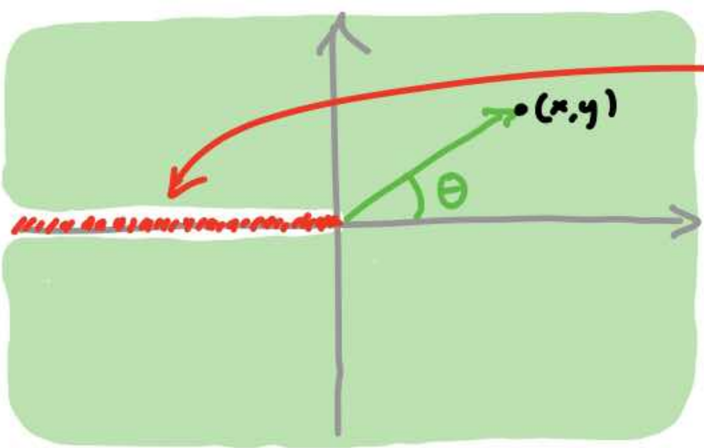
which is also simply connected, i.e. has no holes:

The problem with a <u>non</u>-simply connected set is that the formula for h can develop a discontinuity to one side of a hole,



so that our "$h+G$", whose gradient was $\vec{F}$ on the hole-less rectangle, no longer works.

<u>Upshot for Example A:</u> On any subset of $\mathcal{S}=\mathbb{R}^2\setminus\{\vec{0}\}$ which is simply connected, $\vec{F}_A$ is conservative: on the right half-plane, ... or even on the slit plane:



removing the negative x-axis prevents the circle about the origin that gave us $\int_{\mathcal{C}} \vec{F}\cdot d\vec{r} = 2\pi$.

If we define $\theta(x,y)$ as shown, with range $(-\pi,\pi)$, then $y/x = \tan\theta \Longrightarrow$

- $\frac{1}{x} = \frac{\partial}{\partial y}\frac{y}{x} = \sec^2\theta\,\frac{\partial\theta}{\partial y} \longrightarrow \frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{x} = \frac{x^2}{x\,r^2} = \frac{x}{r^2}$
- $\frac{-y}{x^2} = \frac{\partial}{\partial x}\frac{y}{x} = \sec^2\theta\,\frac{\partial\theta}{\partial x} \longrightarrow \frac{\partial\theta}{\partial x} = \frac{-y\cos^2\theta}{x^2} = \frac{-y\,x^2}{x^2 r^2} = -\frac{y}{r^2}$

$\Longrightarrow \vec{\nabla}\theta = \left(-y/r^2,\ x/r^2\right) = \vec{F}_A$.

Ex/ Determine whether $\vec{F} = (P, Q)$ with

$$\begin{cases} P = 4x^3 + 9x^2y^2 \\ Q = 6x^3y + 6y^5 \end{cases} \quad \text{is conservative.}$$

If so, find the function φ with $\vec{\nabla}\varphi = \vec{F}$.

First, $P_y = 18x^2y = Q_x$, and $\vec{F}$ is defined on all of $\mathbb{R}^2$, which has no holes. Thus, $\vec{F}$ is conservative.

To find φ, write $P = \varphi_x$ and $Q = \varphi_y$. Antidifferentiating the first with respect to x gives

$$x^4 + 3x^3y^2 + C_1(y) = \varphi(x,y),$$

for SOME function C_1 <u>of</u> y alone.

Antidifferentiating the second w.r.t. y gives

$$3x^3y^2 + y^6 + C_2(x) = \varphi(x,y);$$

so $C_1(y) = y^6$ (and $C_2(x) = x^4$). //

Ex/ Suppose you know that $\vec{F} = (P, Q, R)$ with

$$\begin{cases} P = e^x \cos y + yz \\ Q = xz - e^x \sin y \\ R = xy + 2z \end{cases} \quad \text{is conservative.}$$

Compute $\int_C \vec{F} \cdot d\vec{r}$, where C is some path from $\vec{A} = (0, \pi, 2)$ to $\vec{B} = (1, \frac{\pi}{2}, -3)$.

Conservativity is plausible since $P_y = Q_x$, $P_z = R_x$, and $Q_z = R_y$, while $\mathbb{R}^3$ is simply connected (though we haven't yet stated a result like this).

Now antidifferentiate:

- $\varphi_x = P \xrightarrow[\int dx]{} \varphi = e^x \cos y + xyz + C_1(y,z)$
- $\varphi_y = Q \xrightarrow[\int dy]{} \varphi = e^x \cos y + xyz + C_2(x,z)$
- $\varphi_z = R \xrightarrow[\int dz]{} \varphi = xyz + z^2 + C_3(x,y)$

$\Longrightarrow \varphi = e^x \cos y + xyz + z^2$ (plus a constant)

$$\Longrightarrow \int_C \vec{F}\cdot d\vec{r} = \varphi(\vec{B}) - \varphi(\vec{A}) = \left(9 - \frac{3\pi}{2}\right) - 3$$
$$= 6 - \frac{3\pi}{2}.$$

//

We conclude with the n-dimensional converse to Theorem 2 (which we proved so far when $n=2$ and $\mathcal{S}$ is an open rectangle).

Definition: $\mathcal{S}$ is said to be convex if for every $\vec{x}_0, \vec{x}_1 \in \mathcal{S}$, it contains the line segment with $\vec{x}_0, \vec{x}_1$ as endpoints.

proof next time

Theorem 3: Suppose $\mathcal{S}$ is open & convex, $\vec{F} = (f_1, \dots, f_n) : \mathcal{S} \to \mathbb{R}^n$ is C^1, and $\partial f_i/\partial x_j = \partial f_j / \partial x_i$ ($\forall\, i,j$). Then $\vec{F}$ is conservative.

Actually this is still true for $\mathcal{S}$ simply connected (much more general than convex), but we won't prove that.