Lecture 37: A criterion for conservativity

Recall the two main theorems from Lecture 36: assume

- $\mathcal{S} \subset \mathbb{R}^n$ is a connected open set
- $\vec{F} = (f_1, \dots, f_n) : \mathcal{S} \to \mathbb{R}^n$ is C^1

<u>Theorem 1</u>: The following are equivalent

(a) $\vec{F}$ is a gradient field (on $\mathcal{S}$)

(b) $\vec{F}$ is conservative (on $\mathcal{S}$)

(c) $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 0$ for all <u>closed</u> $\mathcal{C} \subset \mathcal{S}$

<u>Theorem 2</u>: If (a)-(c) hold, then $\frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k}$ $\forall j,k$ (on $\mathcal{S}$).

We also claimed, but did not yet prove, a converse to Thm. 2:

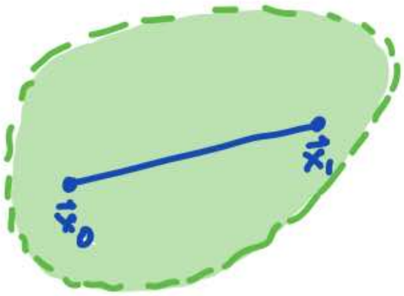
<u>Theorem 3</u>: Suppose $\mathcal{S}$ is convex. Then $\boxed{\frac{\partial f_k}{\partial x_j} = \frac{\partial f_j}{\partial x_k} \ (\forall j,k)} \Rightarrow$ (a)-(c).

(*)

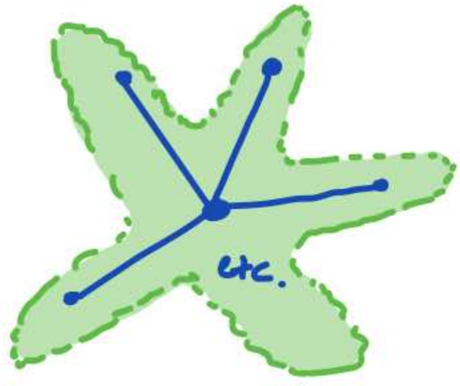
Here <u>convex</u> $\longleftrightarrow$

$\vec{x}_0, \vec{x}_1 \in \mathcal{S} \Rightarrow$

so is the segment



... but the proof actually gives a bit more: $\mathcal{S}$ can be "<u>star-shaped</u>" — containing all segments from a single fixed point to all other points:



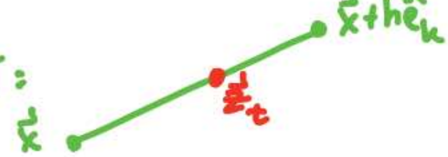
<u>Lemma</u>: Let $R \subset \mathbb{R}^n$ be a rectangle and $\mathcal{J} = R \times [a,b] \subset \mathbb{R}^{n+1}$ (coordinates $(\vec{x}, t)$). Given $\psi : \mathcal{J} \to \mathbb{R}$ with $\frac{\partial \psi}{\partial x_k}$ continuous,

$$\frac{\partial}{\partial x_k} \underbrace{\boxed{\int_a^b \psi(\vec{x}, t)\, dt}}_{=:\varphi(\vec{x})} = \int_a^b \frac{\partial \psi}{\partial x_k}(\vec{x}, t)\, dt .$$

<u>Proof</u>:

$$\frac{\varphi(\vec{x} + h\hat{e}_k) - \varphi(\vec{x})}{h} = \int_a^b \frac{\psi(\vec{x} + h\hat{e}_k) - \psi(\vec{x})}{h}\, dt$$

MVT:



$$= \int_a^b \frac{\partial \psi}{\partial x_k}(\vec{z}_t, t)\, dt$$

$$\Rightarrow \left| \boxed{\frac{\varphi(\vec{x} + h\hat{e}_k) - \varphi(\vec{x})}{h} - \int_a^b \frac{\partial \psi}{\partial x_k}(\vec{x}, t)\, dt} \right| =$$

$$\left| \int_a^b \left(\frac{\partial \psi}{\partial x_k}(\vec{z}_t, t) - \frac{\partial \psi}{\partial x_k}(\vec{x}, t) \right) dt \right| \leq (b-a) \cdot \underbrace{\max_{\begin{cases} t \in [a,b] \\ \vec{z} \in [\vec{x}, \vec{x} + h\hat{e}_k] \end{cases}} \left\{ \left| \frac{\partial \psi}{\partial x_k}(\vec{z}, t) - \frac{\partial \psi}{\partial x_k}(\vec{x}, t) \right| \right\}}_{< \frac{\epsilon}{b-a}}$$

for $0 < |h| < \delta$ suff. small

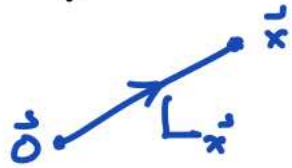
Why? b/c taking $|h| < \delta$ makes $\| (\vec{z}, t) - (\vec{x}, t) \| < \delta$, and $\partial \psi / \partial x_k$ is uniformly continuous on $\mathcal{J}$ by the "small span theorem".

Upshot is that <u>this</u> is arbitrarily small as $h \to 0$, and so (taking its limit)

$$\frac{\partial \varphi}{\partial x_k}(\vec{x}) - \int_a^b \frac{\partial \psi}{\partial x_k}(\vec{x}, t)\, dt = 0 .$$

$\square$

Proof of Theorem 3: We may assume $\vec{0} \in \mathcal{S}$, and that for any $\vec{x} \in \mathcal{S}$ the segment $\vec{r}(t) = t\vec{x}$, $t \in [0,1]$



is too.

$$\text{Set } \varphi(\vec{x}) := \int_{L_{\vec{x}}} \vec{F} \cdot d\vec{r} = \int_0^1 \underbrace{\vec{F}(t\vec{x}) \cdot \vec{x}}_{\psi(\vec{x},t)} \, dt$$

$$\text{Lemma} \Rightarrow \frac{\partial \varphi}{\partial x_k} = \int_0^1 \frac{\partial}{\partial x_k}\left(\vec{F}(t\vec{x}) \cdot \vec{x}\right) dt$$

$$= \int_0^1 \Big(t \underbrace{\frac{\partial \vec{F}}{\partial x_k}}_{\left(\frac{\partial f_1}{\partial x_k}, \dots, \frac{\partial f_n}{\partial x_k}\right) \underset{\text{by } (*)}{=} \left(\frac{\partial f_k}{\partial x_1}, \dots, \frac{\partial f_k}{\partial x_n}\right) = \vec{\nabla} f_k}(t\vec{x}) \cdot \vec{x} + \overbrace{\vec{F}(t\vec{x}) \cdot \hat{e}_k}^{f_k(t\vec{x})} \Big) \, dt$$

$$= \int_0^1 \left(t \vec{\nabla} f_k(t\vec{x}) \cdot \vec{x} + f_k(t\vec{x}) \right) dt$$

$$= \int_0^1 \frac{d}{dt}\left[t f_k(t\vec{x}) \right] dt$$

$$\overset{\text{1-variable}}{\underset{\text{FTC}}{=}} f_k(\vec{x})$$

$$\Rightarrow \vec{\nabla} \varphi = \left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_n} \right) = (f_1, \dots, f_n) = \vec{F}. \qquad \square$$