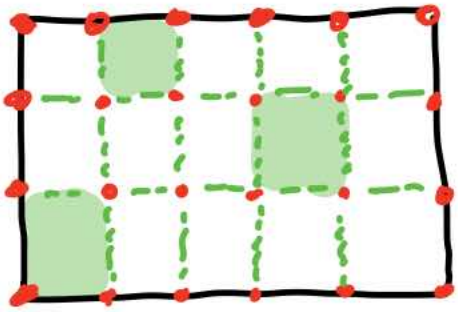
Lecture 38 : Double Integrals

Let's start with step functions.



Q

If $Q = [a,b] \times [c,d]$ and $P_1 = \{x_0, x_1, \dots, x_m\}$, with $a = x_0 < x_1 < \dots < x_m = b$,

$P_2 = \{y_0, y_1, \dots, y_n\}$, with $c = y_0 < y_1 < \dots < y_n = d$, then $P := P_1 \times P_2$ is called a <u>partition of Q</u>. It subdivides Q into little rectangles

$Q_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ with interiors $\mathring{Q}_{ij} = (x_{i-1}, x_i) \times (y_{j-1}, y_j)$.

<u>Definition</u>: $f : Q \to \mathbb{R}$ is a <u>step function</u> if, for some partition P, f is constant on each $\mathring{Q}_{ij}$: $f\big|_{\mathring{Q}_{ij}} \equiv c_{ij}$.

The integral of f is then defined by

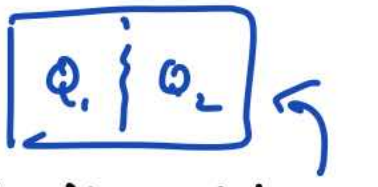
$$\iint_Q f \, dA := \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} (x_i - x_{i-1})(y_j - y_{j-1}).$$

c_{ij} = value of f on $\mathring{Q}_{ij}$; $x_i - x_{i-1} = \Delta x_i$; $y_j - y_{j-1} = \Delta y_j$; $\Delta x_i \Delta y_j$ = area of Q_{ij}

or $\iint_Q f(x,y) \, dx \, dy$

PROPERTIES:

- $\iint_Q (\alpha f + \beta g)\, dA = \alpha \iint_Q f\, dA + \beta \iint_Q g\, dA$
- $\iint_Q f\, dA = \iint_{Q_1} f\, dA + \iint_{Q_2} f\, dA$ if $Q = Q_1 \cup Q_2$ is a subdivision



- $\iint_Q f\, dA \le \iint_Q g\, dA$ if $f \le g$ on Q
- $\iint_Q f\, dA = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$.

Now let $f : Q \to \mathbb{R}$ be a (non-step) function; assume $|f| \le M$ (bounded). Being constant, $-M$ & M are step functions, and $-M \le f \le M$. So

$$S := \left\{ \iint_Q s\, dA \;\middle|\; s \text{ step}, \le f \right\}$$

is bounded above by $\iint_Q M\, dA = M \cdot a(Q)$, while

$$T := \left\{ \iint_Q t\, dA \;\middle|\; t \text{ step}, \ge f \right\}$$

is bounded below by $\iint_Q (-M)\, dA = -M \cdot a(Q)$. Therefore

$$\underline{I} := \sup(S) \quad \text{and} \quad \overline{I} := \inf(T) \quad \text{exist.}$$

(least upper bound; greatest lower bound)

Moreover, since for any given step function $t \ge f$

$$\iint_Q s\, dA \le \iint_Q t\, dA \quad \text{for all step fns. } s \le f,$$

we have $\underline{I} \le \iint_Q t\, dA$ (for any $t \ge f$), hence $\underline{I} \le \overline{I}$.

Definition: If $\underline{I} = \overline{I}$ then f is integrable on Q, and $\iint_Q f\,dA$ is defined to be this common value.

HOW TO COMPUTE IT?!

Fubini's Theorem: If

- $f: Q \to \mathbb{R}$ is integrable on Q,
- $f(x, y_0)$ is integrable on $[a,b]$ for each $y_0 \in [c,d]$, and
- $A(y) := \int_a^b f(x,y)\,dx$ is integrable on $[c,d]$,

then $\iint_Q f\,dA = \int_c^d A(y)\,dy$. (Obviously we can also reverse the roles of x & y.)

$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$ is called an iterated integral.

Proof: Let $s \le f \le t$, s & t step functions. For each y,

$$\int_a^b s(x,y)\,dx \le \int_a^b f(x,y)\,dx \le \int_a^b t(x,y)\,dx, \quad \text{hence}$$

$$\iint_Q s\,dA = \int_c^d \int_a^b s\,dx\,dy \le \int_c^d A(y)\,dy \le \int_c^d \int_a^b t\,dx\,dy = \iint_Q t\,dA$$

Since s & t are arbitrary,

$$\underline{I} \le \int_c^d A(y)\,dy \le \overline{I}$$

$$= \iint_Q f\,dA \text{ by assumption.}$$

□

Ex 1/ Compute $I = \int_0^{\sqrt{3}} \left(\int_0^1 \frac{8x}{(x^2+y^2+1)^2}\, dy \right) dx$.

The inner integral is best avoided; using Fubini twice,

$$I = \iint_{[0,\sqrt{3}]\times[0,1]} \frac{8x}{(x^2+y^2+1)^2}\, dA = \int_0^1 \left(\int_0^{\sqrt{3}} \frac{8x}{(x^2+y^2+1)^2}\, dx \right) dy$$

$$= \int_0^1 \left[\frac{-4}{x^2+y^2+1} \right]_{x=0}^{\sqrt{3}} dy = \int_0^1 \left(\underbrace{\frac{-4}{4+y^2}}_{\frac{-1}{1+(y/2)^2}} + \frac{4}{1+y^2} \right) dy$$

$$= \left[-2\arctan\left(\frac{y}{2}\right) + 4\arctan(y) \right]_0^1 = \left(-2\arctan\left(\tfrac{1}{2}\right) + 4\arctan(1)\right) - (0+0)$$

$$= \pi - 2\arctan(1/2). \qquad //$$

Ex 2/ Find $I := \iint_Q y\sin(xy)\, dA$, where $Q = [1,2]\times[0,\pi]$.

$$I = \int_1^2 \left(\int_0^{\pi} y\sin(xy)\, dy \right) dx = \int_1^2 \left[-\frac{y}{x}\cos(xy)\Big|_{y=0}^{\pi} + \int_0^{\pi} \frac{1}{x}\cos(xy)\, dy \right] dx$$

(by parts: $u = y$, $du = dy$; $dv = \sin(xy)\,dy$, $v = -\frac{1}{x}\cos(xy)$)

$$= \int_1^2 \left[-\frac{\pi}{x}\cos(\pi x) + \frac{1}{x^2}\sin(\pi x) \right] dx = -\frac{1}{x}\sin(\pi x)\Big|_1^2 - \int_1^2 \frac{1}{x^2}\sin(\pi x)\, dx + \int_1^2 \frac{1}{x^2}\sin(\pi x)\, dx$$

(by parts on 1st term only: $u = -\frac{1}{x}$, $dv = \pi\cos(\pi x)\,dx$, $du = dx/x^2$, $v = \sin(\pi x)$)

$$= -\frac{1}{x}\sin(\pi x)\Big|_1^2 = -\frac{1}{2}\sin 2\pi + \sin\pi = 0.$$

A lot of work for that 0... alternatively:

$$I = \int_0^{\pi} \left(\int_1^2 y\sin(xy)\, dx \right) dy = \int_0^{\pi} \left[-\cos(xy) \right]_{x=1}^{2} dy$$

$$= \int_0^{\pi} \left(\cos y - \cos 2y \right) dy = \left[\sin y - \tfrac{1}{2}\sin 2y \right]_0^{\pi} = 0$$

Moral: the order you choose can make a big difference (for the work, not the answer). //

Sometimes $f(x,y)$ breaks up naturally into $F(x)\,G(y)$. For example, $\frac{y}{1+x^2}$ or $xye^{x^2+y^2}\ (= xe^{x^2}\cdot ye^{y^2})$. In this case, the double integral itself factors:

$$\iint_Q f(x,y)\,dA = \int_a^b \left(\int_c^d \underbrace{F(x)}_{\text{constant w.r.t. } y}\, G(y)\,dy \right) dx$$

$$= \int_a^b \left(F(x) \underbrace{\int_c^d G(y)\,dy}_{\text{constant w.r.t. } x} \right) dx$$

$$= \left(\int_c^d G(y)\,dy \right)\left(\int_a^b F(x)\,dx \right).$$

Ex 3/ $$\int_{[0,1]^2} xye^{x^2+y^2}\,dA = \left(\int_0^1 xe^{x^2}dx\right)^2 = \left(\left[\tfrac{1}{2}e^{x^2}\right]_0^1\right)$$

$$= \left(\tfrac{1}{2}(e-1)\right)^2 = \frac{(e-1)^2}{4}. \qquad //$$

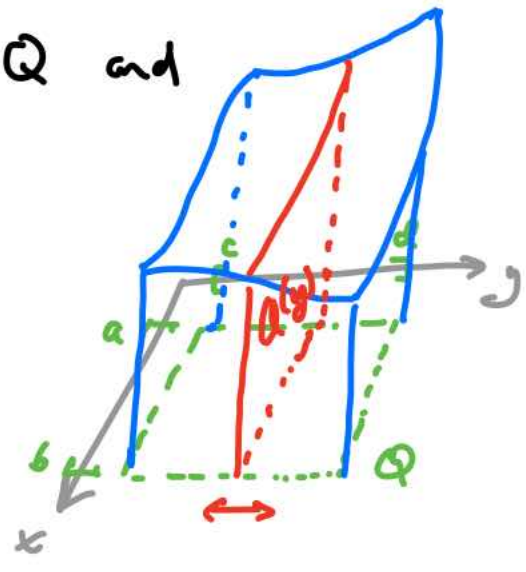
Fubini's Theorem also allows us to relate double integrals to volume. By Lecture 10 of last semester, the integral of (measurable) cross-sections of a solid ($\perp$ to a fixed line) gives its volume. So if $f \geq 0$ on Q and

$$S := \{(x,y,z) \mid (x,y)\in Q,\ 0 \leq z \leq f(x,y)\}$$

then the volume of S is

$$v(S) = \int_c^d A(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$$

$$= \iint_Q f\,dA.$$



Ex 4/ Find the volume of the solid bounded by the surfaces $z = \frac{y}{(xy+1)^2}$ and the planes $z=0$, $x=0$, $x=1$, $y=0$, and $y=1$.

which way to integrate first?

$$V = \iint_{[0,1]^2} \frac{y}{(xy+1)^2}\,dA = \int_0^1 \left(\int_0^1 \frac{y}{(xy+1)^2}\,dx \right) dy$$

$$= \int_0^1 \left[\frac{-1}{xy+1} \right]_{x=0}^{1} dy = \int_0^1 \left(1 - \frac{1}{1+y} \right) dy$$

$$= \left[y - \ln(1+y) \right]_0^1 = 1 - \ln(2). \qquad //$$

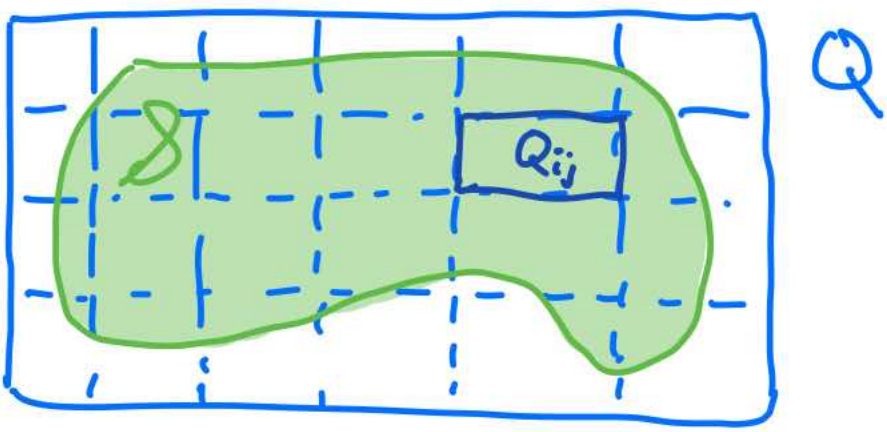
It's also worth a look at Apostol's Ex. 2 in § 11.8.

Applications

(1) Mass

A lamina is a flat sheet so thin we can think of it as 2-D; made of non-homogenous material (density can vary).

If a lamina covers a region $S \subset \mathbb{R}^2$ with mass density function $\delta : S \to \mathbb{R}$, we may extend δ to Q by defining it to be zero off S.

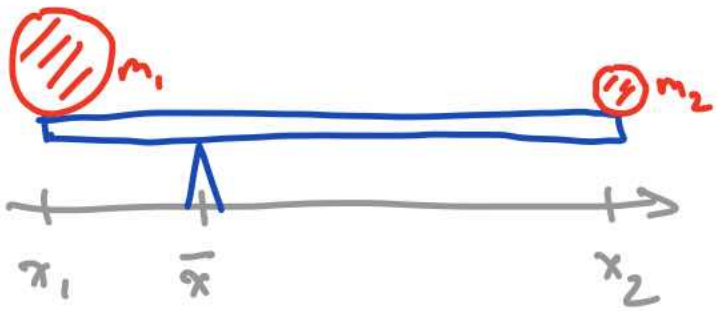


Pick representative points $q_{ij} = (x_{ij}, y_{ij}) \in Q_{ij}$. Then the mass $m(Q_{ij}) \approx \delta(q_{ij})\, a(Q_{ij}) \Rightarrow$ total mass $m(S) = m(Q) \approx \sum_{i,j} \delta(q_{ij})\, a(Q_{ij})$.

By the small span theorem, taking the size of the partition to 0 makes the integrals of the upper & lower step functions coincide, so

$$m = \iint_S \delta \, dA \quad \text{(is the mass of the lamina)}.$$

(2) Center of mass Now let's think about seesaws:

for this 

to balance,

we need $m_1(\bar{x} - x_1) = m_2(x_2 - \bar{x})$ (equality of moments).

Solving for $\bar{x}$ gives $(m_1 + m_2)\bar{x} = m_1 x_1 + m_2 x_2 \Rightarrow$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

← argue by induction

For n masses, the balance point would be

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \quad \left(\text{if in 2-D, } \bar{y} = \frac{\sum y_i m_i}{\sum m_i}\right)$$

Thinking of the lamina as a bunch of small masses (at q_{ij}) yields

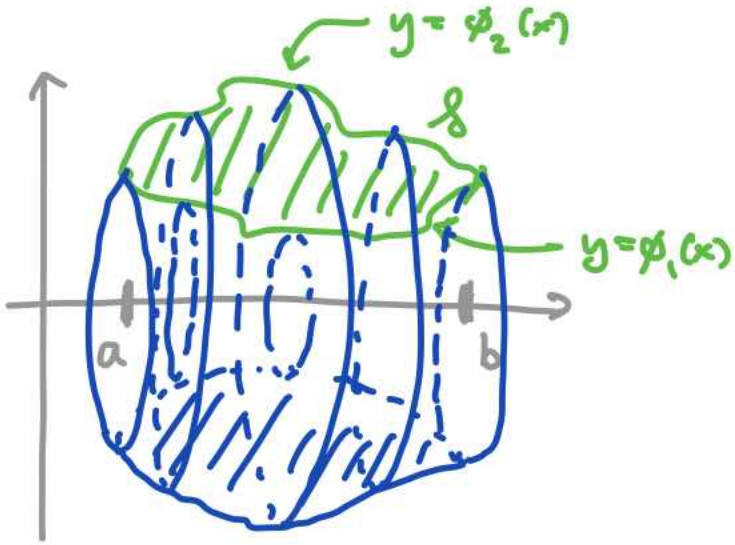
$$\bar{x} \approx \frac{\sum_i \sum_j x_{ij}\, \delta(q_{ij})\, a(Q_{ij})}{\sum_i \sum_j \delta(q_{ij})\, a(Q_{ij})} \quad \text{etc.}$$

hence (taking the limit)

$$\bar{x} = \frac{\iint_{\mathcal{S}} x\,\delta(x,y)\,dA}{\iint_{\mathcal{S}} \delta(x,y)\,dA} = \frac{M_y}{m}, \qquad \bar{y} = \frac{\iint_{\mathcal{S}} y\,\delta(x,y)\,dA}{\iint_{\mathcal{S}} \delta(x,y)\,dA} = \frac{M_x}{m}$$

where M_x & M_y are the moments about the x- and y-axes, and $(\bar{x},\bar{y})$ is the center of mass or centroid of $\mathcal{S}$.

(3) Pappus's Theorem



Suppose we now rotate $\mathcal{S}$ about the x-axis, so that the centroid traces out a circle of radius $\bar{y}$. The resulting solid has volume

$$V = \int_a^b \pi\left(\phi_2(x)^2 - \phi_1(x)^2\right)dx$$

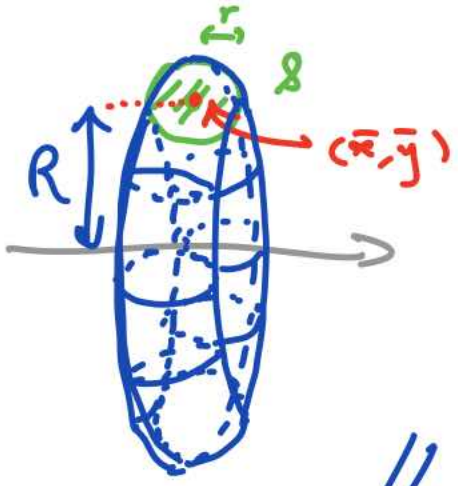
$$= \pi\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} 2y\,dy\right)dx = 2\pi\iint_{\mathcal{S}} y\,dA$$

$$= 2\pi M_x = 2\pi\bar{y}\iint_{\mathcal{S}} dA$$

$$= 2\pi\bar{y}\,a(\mathcal{S}),$$

where the mass density $\delta \equiv 1$. (See Apostol for another theorem in this vein, concerning the centroid of a disjoint union $\mathcal{S}_1 \cup \mathcal{S}_2$.)

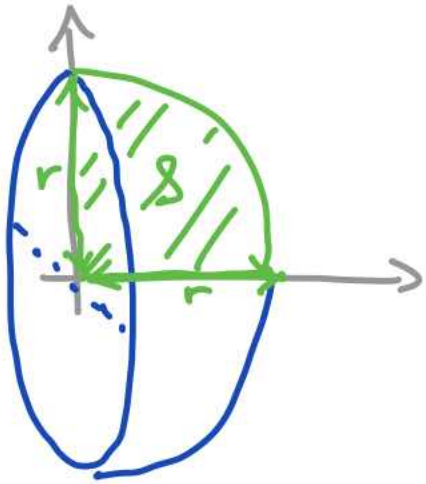
Ex 5/ Find the volume of the torus T depicted:



$$V(T) = 2\pi \bar{y}\, a(\delta) = 2\pi R \cdot \pi r^2$$
$$= 2\pi^2 r^2 R$$

//

Ex 6/ Find the centroid of the quarter-circle shown:



Rotating about the x-axis yields a hemisphere of radius r and volume

we'll actually check this later

$$\frac{2}{3}\pi r^3 = V = 2\pi \bar{y}\, \underbrace{a(\delta)}_{\frac{1}{4}\pi r^2} = \frac{\pi^2}{2} r^2 \bar{y}$$

Pappus

$$\Rightarrow \quad \bar{x} = \bar{y} = \frac{4r}{3\pi}.$$

//