Lecture 38

Non-rectangular Double Integrals
Let $S$ be a closed, bounded subset of $\mathbb{R}^2$, and enclose it in a closed rectangle $Q$.
Let $\mathcal{S}$ be a closed, bounded subset of $\mathbb{R}^2$, and enclose it in a closed rectangle $\mathcal{Q}$:

If $f(x,y)$ is a function on $\mathcal{S}$, define a function on $\mathcal{Q}$ by

$$
\tilde{f}(x,y) = \begin{cases} 
  f(x,y) & \text{if } (x,y) \in \mathcal{S} \\
  0 & \text{if } (x,y) \in \mathcal{Q}\setminus\mathcal{S}
\end{cases}
$$
Let $\mathcal{Q}$ be a closed, bounded subset of $\mathbb{R}^2$, and enclose it in a closed rectangle $\mathcal{Q}$:

If $f(x,y)$ is a function on $\mathcal{Q}$, define a function on $\mathcal{Q}$ by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in \mathcal{Q} \\ 0 & \text{if } (x,y) \in \mathcal{Q} \setminus \mathcal{Q} \end{cases}$$

**Definition:** $f$ is integrable (on $\mathcal{Q}$) $\iff$ $\tilde{f}$ is integrable (on $\mathcal{Q}$). In this case

$$\iint_{\mathcal{Q}} f \, dA := \iint_{\mathcal{Q}} \tilde{f} \, dA.$$
Let $S$ be a closed, bounded subset of $\mathbb{R}^2$, and enclose it in a closed rectangle $Q$:

If $f(x,y)$ is a function on $S$, define a function on $Q$ by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in S \\ 0 & \text{if } (x,y) \in Q \setminus S \end{cases}$$

**Definition:** $f$ is integrable (on $S$) $\iff$ $\tilde{f}$ is integrable (on $Q$). In this case

$$\iint_S f \, dA := \iint_Q \tilde{f} \, dA.$$ 

We shall see later that if $S$ has piecewise smooth boundary and $f$ is piecewise continuous on $S$, it is integrable.

Meanwhile, how do we evaluate double integrals on $S$?
Let $\mathcal{S}$ be a closed, bounded subset of $\mathbb{R}^2$, and enclose it in a closed rectangle $Q$:

If $f(x,y)$ is a function on $\mathcal{S}$, define a function on $Q$ by

$$\tilde{f}(x,y) := \begin{cases} f(x,y) & \text{if } (x,y) \in \mathcal{S} \\ 0 & \text{if } (x,y) \in Q \setminus \mathcal{S} \end{cases}$$

Definition: $f$ is integrable (on $\mathcal{S}$) $\iff$ $\tilde{f}$ is integrable (on $Q$). In this case

$$\iint f \, dA := \iint \tilde{f} \, dA.$$ 

We shall see later that if $\mathcal{S}$ has piecewise smooth boundary and $f$ is piecewise continuous on $\mathcal{S}$, it is integrable.

Meanwhile, how do we evaluate double integrals on $\mathcal{S}$?

Definition: $\mathcal{S}$ is of type I if each line parallel to the $y$-axis intersects $\mathcal{S}$ in a single closed interval (or a point, or not at all).
Definition: $f$ is integrable (on $S$) $\iff$ 
$f$ is integrable (on $\Omega$). In this case 
$$\iint_S f \, d\lambda = \iint_\Omega \tilde{f} \, d\lambda.$$

We shall see later that if $S$ has piecewise smooth boundary and $f$ is piecewise continuous on $S$, it is integrable.

Meanwhile, how do we evaluate double integrals on $S$?

Definition: $S$ is of type II if each line parallel to the $x$-axis intersects $S$ in a single closed interval (or a point, or not at all).
So a type I set takes the form
\[ S = \{(x,y) | a \leq x \leq b, \phi(x) \leq y \leq \phi_2(x)\} \]
for some functions \( \phi, \phi_2 : [a,b] \to \mathbb{R} \).

**Definition:** \( f \) is integrable (on \( S \)) \( \iff \)
\( \tilde{f} \) is integrable (on \( B \)). In this case
\[ \iint_S f \, dA = \iint_B \tilde{f} \, dA. \]

We shall see later that if \( S \) has piecewise smooth boundary and \( f \) is piecewise continuous on \( S \), it is integrable.

Meanwhile, how do we evaluate double integrals on \( S \)?

**Definition:** \( S \) is of type I if each line parallel to the \( y \)-axis intersects \( S \) in a single closed interval (or a point, or not at all).
So a type I set takes the form

\[ S = \{ (x, y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x) \} \]

for some functions \( \phi_1, \phi_2 : [a, b] \to \mathbb{R} \).

Enclosing it in \( Q = (a, b) \times (c, d) \),

\[ \iint_S f \, dA = \iint_Q f \, dA = \int_{a}^{b} \left( \int_{c}^{d} f(x, y) \, dy \right) \, dx \]

\( \text{Fubini} \)

Definition: \( f \) is integrable (on \( S \)) \iff \( \tilde{f} \) is integrable (on \( Q \)). In this case

\[ \iint_S f \, dA = \iint_Q \tilde{f} \, dA. \]

We shall see later that if \( S \) has piecewise smooth boundary and \( f \) is piecewise continuous on \( S \), it is integrable.

Meanwhile, how do we evaluate double integrals on \( S \)?

Definition: \( S \) is of type I if each line parallel to the y-axis intersects \( S \) in a single closed interval (or a point, or not at all).
So a type I set takes the form
\[ A = \{ (x,y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x) \} \]
for some functions \( \phi_1, \phi_2 : [a,b] \to \mathbb{R} \).

Enclosing it in \( Q = (a,b) \times [c,d] \),
\[
\iint_Q f \, dA = \iint_A f \, dA = \int_a^b \left( \int_c^d \tilde{f}(x,y) \, dy \right) \, dx.
\]

Fubini:
\[
= \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx.
\]

Definition: \( f \) is integrable (on \( A \)) \( \iff \)
\( \tilde{f} \) is integrable (on \( Q \)). In this case
\[
\iint_A f \, dA := \iint_Q \tilde{f} \, dA.
\]

We shall see later that if \( A \) has piecewise smooth boundary and \( f \) is piecewise continuous on \( A \), it is integrable.

Meanwhile, how do we evaluate double integrals on \( A \)?

Definition: \( A \) is of type I if each line parallel to the y-axis intersects \( A \) in a single closed interval (or a point, or not at all).
So a type I set takes the form
\[ S = \{ (x,y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x) \} \]
for some functions \( \phi_1, \phi_2 : [a,b] \to \mathbb{R} \).

Enclosing it in \( Q = (a,b) \times [c,d] \),
\[
\iint_S f \, dA = \iint_Q \tilde{f} \, dA = \int_c^d \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y) \, dy \right) \, dx
\]
\[= \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx \quad \text{[Fubini]} \]

Similarly, a type II set may be written
\[ S = \{ (x,y) \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y) \}, \]
and
\[
\iiint_S f \, dA = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x,y) \, dx \right) \, dy.
\]

**Definition:** \( f \) is integrable (on \( S \)) \( \iff \) \( \tilde{f} \) is integrable (on \( Q \)). In this case
\[
\iint_S f \, dA := \iint_Q \tilde{f} \, dA.
\]

We shall see later that if \( S \) has piecewise smooth boundary and \( f \) is piecewise continuous on \( S \), it is integrable.

Meanwhile, how do we evaluate double integrals on \( S \)?

**Definition:** \( S \) is of type \( \text{II} \) if each

the \( x \)-axis intersects \( S \) in a single closed interval (or a point, or not at all).
So a type I set takes the form

\[ D = \{ (x, y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x) \} \]

for some functions \( \phi_1, \phi_2 : [a, b] \to \mathbb{R} \).

Enclosing it in \( Q = [a, b] \times [c, d] \),

\[ \iint_{Q} f \, dA = \iint_{D} f \, dA = \int_{a}^{b} \left( \int_{c}^{d} f(x, y) \, dy \right) \, dx \]

Using Fubini,

\[ = \int_{a}^{b} \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) \, dy \right) \, dx \]

Similarly, a type II set may be written

\[ D = \{ (x, y) \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y) \} \]

and

\[ \iint_{D} f \, dA = \int_{c}^{d} \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) \, dx \right) \, dy \].
So a type I set takes the form

\[ S = \{(x,y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\} \]

for some functions \( \phi_1, \phi_2 : [a,b] \to \mathbb{R} \). Enclosing it in \( Q = (a,b) \times [c,d] \),

\[
\iint_S f \, dA = \iint_Q f \, dA = \int_a^b \left( \int_c^d f(x,y) \, dy \right) \, dx
\]

[Fubini]

\[
= \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y) \, dy \right) \, dx
\]

\[
= \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y) \, dy \right) \, dx
\]

\[
\text{Similarly, a type II set may be written}
\]

\[ S = \{(x,y) \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\}, \]

and

\[
\iint_S f \, dA = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x,y) \, dx \right) \, dy
\]

\[ S \text{ is both of type I & type II, but the functions for the type II integration are easier (and already written):}
\]

\[ \iint_S (y+1) \, dA = \int_0^1 \left( \int_{y^2-4}^{y^2+4} (y+1) \, dx \right) \, dy \]
So a type I set takes the form

\[ S = \{(x,y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\} \]

for some functions \( \phi_1, \phi_2 : [a,b] \to \mathbb{R} \).

Enclosing it in \( Q = [a,b] \times [c,d] \),

\[ \iiint_Q f \, dV = \iint_S f \, dA = \int_a^b \left( \int_c^{\phi_2(x)} f(x,y) \, dy \right) \, dx \]

\[ = \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x,y) \, dy \right) \, dx \]

\( \text{Fubini} \)

Ex 1] Compute \( \iint_S (y+1) \, dA \)

\( S \) is both of type I & type II, but the functions for the type II
integration are easier (and already written):

\[ \iiint_S (y+1) \, dV = \int_0^1 \left( \int_{y^2-4}^{y^2+y+4} (y+1) \, dx \right) \, dy \]

\[ = \int_0^1 \left[ x(y+1) \right]_{x=y^2-4}^{x=y^2+y+4} \, dy \]

\[ = \int_0^1 \left( y^2+y+4 - (y^2-4) \right) (y+1) \, dy \]

Similarly, a type II set may be written

\[ S = \{(x,y) \mid c \leq y \leq d, \psi_1(y) \leq x \leq \psi_2(y)\} \],

and

\[ \iint_S f \, dV = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x,y) \, dx \right) \, dy \].
Ex 1) Compute \( \iint_\mathcal{D} (y+1) \, dA \)

\( \mathcal{D} \) is both of type I & type II, but the functions for the type II integration are easier (and already written):

\[
\iint_\mathcal{D} (y+1) \, dA = \int_0^1 \left( \int_{y^2-4}^{y^2+4} (y+1) \, dx \right) \, dy
\]

\[
= \int_0^1 \left[ x(y+1) \right]_{x=y^2-4}^{x=y^2+4} \, dy
\]

\[
= \int_0^1 (-y^2+4 - (y^2-4))(y+1) \, dy
\]

\[
= \cdots = \frac{65}{6}
\]
Ex 2*] Find \( \iiint_{S} (8x + 10y)\,dA \), where

\( S \) is the region between the graphs of \( y = x^2 \) and \( y = 2x \).

Ex 1] Compute \( \iiint_{S} (y+1)\,dA \)

\( S \) is both of type I & type II, but the functions for the type II integration are easier (and already written):

\[
\iiint_{S} (y+1)\,dA = \int_{0}^{1} \left( \int_{y^2-4}^{-y^2+4} (y+1)\,dx \right)\,dy
\]

\[
= \int_{0}^{1} \left[ x(y+1) \right]_{x=y^2-4}^{x=-y^2+4} \,dy
\]

\[
= \int_{0}^{1} (-y^2+4 - (y^2-4))(y+1)\,dy
\]

\[
= \cdots = \frac{65}{6}
\]
Ex 1] Compute \( \iint_{R} (y+1) \, dA \)

\( R \) is both of type I & type II, but the functions for the type II integration are easier (and already written):

\[
\iint_{R} (y+1) \, dA = \int_{0}^{1} \left( \int_{y^{2}-4}^{-y^{2}+4} (y+1) \, dx \right) \, dy
\]

\[
= \int_{0}^{1} \left[ x(y+1) \right]_{x=y^{2}-4}^{x=-y^{2}+4} \, dy
\]

\[
= \int_{0}^{1} (-y^{2}+4 - (y^{2}-4)) (y+1) \, dy
\]

\[
= \ldots = \frac{65}{6}
\]
Ex 2. Find \( \iint_{S} (8x + 10y) \, dA \), where

\( S \) is the region between the graphs of \( y = x^2 \) and \( y = 2x \).

\[
\iint_{S} (8x + 10y) \, dA = \int_{0}^{2} \left( \int_{x^2}^{2x} (8x + 10y) \, dy \right) \, dx
\]

\[
= \int_{0}^{2} \left[ 8xy + 5y^2 \right]_{y=x^2}^{2x} \, dx
\]

\[
= \int_{0}^{2} \left( 16x^2 + 20x - (8x^3 + 5x^4) \right) \, dx
\]

\[
= \int_{0}^{2} \left( -5x^4 - 8x^3 + 36x^2 \right) \, dx
\]

\[
= \ldots = 32.
\]
Ex 2

Find \( \int_S (8x+10y) \, dA \), where

\( S \) is the region between the graphs of \( y = x^2 \) and \( y = 2x \).

\[
\int_S = \int_0^2 \left( \int_{x^2}^{2x} (8x+10y) \, dy \right) \, dx
\]

\[
= \int_0^2 \left[ 8xy + 5y^2 \right]_{y=x^2}^{2x} \, dx
\]

\[
= \int_0^2 \left( (16x^2+20x) - (8x^3+5x^4) \right) \, dx
\]

\[
= \int_0^2 (-5x^4 - 8x^3 + 36x^2) \, dx
\]

\[
= \ldots = 32.
\]

What if \( S \) is more complicated?

(\textit{i.e.} neither type I or type II)
Ex 2* Find \( \iint_{\mathcal{D}} (8x+10y) \, dA \), where \( \mathcal{D} \) is the region between the graphs of \( y = x^2 \) and \( y = 2x \).

\[
\iint_{\mathcal{D}} = \int_{0}^{2} \left( \int_{x^2}^{2x} (8x+10y) \, dy \right) \, dx
\]

\[
= \int_{0}^{2} \left( 8xy + 5y^2 \right)_{y=x^2}^{2x} \, dx
\]

\[
= \int_{0}^{2} \left( (16x^2 + 20x^2) - (8x^3 + 5x^4) \right) \, dx
\]

\[
= \int_{0}^{2} (-5x^4 - 8x^3 + 36x^2) \, dx
\]

\[
= \ldots = 32.
\]
What if \( S \) is more complicated?

\[ S = S_I \cup S_{II} \]

(and \( S_I \cap S_{II} \) a curve)

Split it into subsets and write

\[ \iiint_S = \iiint_{S_I} + \iiint_{S_{II}} \]

More generally, you can use this technique (breaking the region) to compute integrals which are of type I or type II:

Ex 3 / Find \( \iiint_S 1 \, dA \), where \( S \) is in the 1st quadrant, bounded by \( y = x^2 \), \( y = \frac{x^2}{8} \), and \( y = \frac{1}{x} \).
What if $\Omega$ is more complicated?

\[
\Omega = \Omega_1 \cup \Omega_2
\]

(and $\Omega_1 \cap \Omega_2$ a curve)

Split it into subsets and write

\[
\iint_{\Omega} f = \iint_{\Omega_1} f + \iint_{\Omega_2} f.
\]

More generally, you can use this technique (breaking the region) to compute integrals which are of type I or type II.

Ex 3 / Find $\iint_{\Omega} 1 \, dA$, where $\Omega$ is in the 1st quadrant, bounded by $y = x^2$, $y = \frac{x^2}{8}$, and $y = \frac{1}{x}$. 
Draw it:

Divide it:

\[ \iint_A 1 \, dA = \iint_{A_1} 1 \, dA + \iint_{A_2} 1 \, dA \]

What if \( A \) is more complicated?

\[ A = A_1 \cup A_2 \] (and \( A_1 \cap A_2 \) a curve)

Split it into subsets and write

\[ \iint_A = \iint_{A_1} + \iint_{A_2} \]

More generally, you can use this technique (breaking the region) to compute integrals which are of type I or type II:

Ex 3: Find \( \iint_A 1 \, dA \), where \( A \) is in the 1st quadrant, bounded by \( y = x^2 \), \( y = \frac{x^2}{8} \), and \( y = \frac{1}{x} \).
What if \( S \) is more complicated?

\[
S = S_I \cup S_{II}
\]
(and \( S_I \cap S_{II} \) a curve)

Split it into subsets and write

\[
\iint_S = \iint_{S_I} + \iint_{S_{II}}
\]

More generally, you can use this technique (breaking the region) to compute integrals which are of type I or type II:

Ex 3 / Find \( \iint_S 1 \, dA \), where \( S \) is in the 1st quadrant, bounded by \( y = x^2, \ y = \frac{x^2}{8}, \) and \( y = \frac{1}{x} \).
\[
\int_{x}^{1} \frac{1}{y} \, dy = \int_{x}^{1} \frac{1}{y} \, dy + \int_{x}^{1} \frac{1}{y} \, dy
\]
(by the way, this computes area of \(8\))

\[
= \int_{0}^{1} \int_{x}^{x^2} 1 \, dy \, dx + \int_{1}^{2} \int_{x}^{x^2} 1 \, dy \, dx
\]

\[
= \int_{0}^{1} (x^2 - \frac{x^2}{8}) \, dx + \int_{1}^{2} (\frac{1}{x} - \frac{x^2}{8}) \, dx
\]

\[
= ... = \ln (2)
\]
Draw it:
Divide it:

\[ \iint_S 1 \, dA = \iint_{S'} 1 \, dA + \iint_{S''} 1 \, dA \]
(by the way, this computes area of S)
\[ = \int_0^1 \int_{0}^{x^2} 1 \, dy \, dx + \int_1^2 \int_{x^2/8}^{y} 1 \, dy \, dx \]
\[ = \int_0^1 (x^2 - \frac{x^2}{8}) \, dx + \int_1^2 (\frac{1}{x} - \frac{x^2}{8}) \, dx \]
\[ = ... = \ln(2) \]

Now for something trickier.
Ex 4* Find \( \int_0^4 \left( \int_{x/2}^{x^2} e^{y^2} \, dy \right) \, dx \).

Well, of course you can’t differentiate \( e^{y^2} \).

But you also can’t just “exchange the integrals” — you’d end up with our limits that aren’t constants, which makes no sense.

\[
\begin{align*}
\int_{x/2}^{x^2} \int_1^1 1 \, dy \, dx + \int_{x^2}^{x^2} \int_{x^2}^{x^2} 1 \, dy \, dx \\
= \int_{x/2}^{x^2} \int_{x^2}^{x^2} 1 \, dy \, dx + \int_{x^2}^{x^2} \int_{x^2}^{x^2} 1 \, dy \, dx \\
= \int_0^1 \left( x^2 - \frac{x^2}{8} \right) \, dx + \int_1^2 \left( \frac{1}{x} - \frac{x^2}{8} \right) \, dx \\
= \cdots = \ln(2)
\end{align*}
\]

Now for something trickier.
Ex 4* Find \( \int_0^4 \left( \int_{x/2}^{y^2} e^{y^2} \, dy \right) \, dx \).

Well, of course you can't differentiate \( e^{y^2} \). But you also can't just "exchange the integrals" — you'd end up with your limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration:

\[
\int_0^1 \int_{x^2/4}^{2} e^{x^2 \frac{y}{2}} \, dy \, dx = \int_0^1 \left( x^2 - \frac{x^4}{8} \right) \, dx + \int_1^2 \left( \frac{1}{x} - \frac{x^2}{8} \right) \, dx
\]

\[= \ln(2) \]

Now for something trickier.
Draw it:

Divide it:

\[ \int_0 \frac{1}{x} \, dA = \int_{x_1} \frac{1}{x} \, dA + \int_{x_2} \frac{1}{x} \, dA \]
(by the way, this computes area of \( A \))

\[ = \int_0^1 \int_{x_1} x^2 \, dy \, dx + \int_1^2 \int_{x_2} x^2 \, dy \, dx \]
\[ = \int_0^1 (x^2 - \frac{x^2}{8}) \, dx + \int_1^2 (\frac{1}{x} - \frac{x^2}{8}) \, dx \]
\[ = \cdots = \ln (2) \]

Ex 4* Find \( \int_0^4 \left( \int_{x/2} e^{y^2} \, dy \right) \, dx \).

Well, of course you can't differentiate \( e^{y^2} \).

But you also can't just "exchange the integrals" — you'd end up with outer limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variable of integration:

The integral above is

\[ \int_0^2 \left( \int_0^2 e^{y^2} \, dy \right) \, dx \]

(b/c \( A = \{(x,y) \mid 0 \leq x \leq 4, \frac{x}{2} \leq y \leq 2 \} \)

\[ = \{(x,y) \mid 0 \leq y \leq 2, 0 \leq x \leq 2y \} \)

Now for something trickier.
Ex 4* Find \( \int_0^4 \left( \int_{x/2}^2 e^{y^2} \, dy \right) \, dx \).

Well, of course you can't differentiate \( e^{y^2} \).

But you also can't just "exchange the integrals" — you'd end up with outer limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration:

The integral above is:

\[
SS_2 e^{y^2} \, dA = \int_1^2 \left( \int_0^{2y} e^{y^2} \, dx \right) \, dy
\]

\( b/c \ R = \{(x, y) \mid 0 \leq x \leq 4, \ \frac{x}{2} \leq y \leq 2 \}
= \{(x, y) \mid 0 \leq y \leq 2, \ 0 \leq x \leq 2y \}
= \int_0^2 2y e^{y^2} \, dy = \left[ e^{y^2} \right]_0^2 = e^4 - 1. \)
Ex 5 Determine the volume of the tetrahedron bounded by the coordinate planes and the plane $3x + 6y + 4z = -12 = 0$.

Ex 4* Find $\int_0^4 \left( \int_{x/2}^{2y} e^{y^2} \, dy \right) \, dx$.

Well, of course you can't differentiate $e^{y^2}$. But you also can't just "exchange the integrals" — you'd end up with our limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration:

The integral above =

$$\iiint_8 e^{y^2} \, dA = \int_0^2 \left( \int_0^{2y} e^{y^2} \, dx \right) \, dy$$

(b/c $\mathcal{D} = \{(x,y) \mid 0 \leq x \leq 4, \, \frac{x}{2} \leq y \leq 2\} = \{(x,y) \mid 0 \leq y \leq 2, \, 0 \leq x \leq 2y\}$)

$$= \int_0^2 2y e^{y^2} \, dy = [e^{y^2}]_0^2 = e^4 - 1.$$
Ex 5] Determine the volume of the tetrahedron bounded by the coordinate planes and the plane $3x + 6y + 4z = -12 = 0$.

Let $\mathcal{S}$ be the triangular region in the $xy$-plane under the base of this tetrahedron.

(Setting $z=0$ in the equation gives $3x + 6y = 12$ or $y = 2 - \frac{1}{2}x$ for this boundary curve.)

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Ex 4*] Find $\int_0^4 \left( \int_{x/2}^{y^2} e^{y^2} \, dx \right) \, dy$.

Well, of course you can't differentiate $e^{y^2}$.

But you also can't just "exchange the integrals"—you'd end up with outer limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration:

The integral above is

$$
\iint_{\mathcal{S}} e^{y^2} \, dA = \int_0^2 \left( \int_{x/2}^{y^2} e^{y^2} \, dx \right) \, dy
$$

(b/c $\mathcal{S} = \{(x,y) \mid 0 \leq x \leq 4, \frac{x}{2} \leq y \leq 2 \}$

$= \{(x,y) \mid 0 \leq y \leq 2, 0 \leq x \leq 2y \}$

$= \int_0^2 2y e^{y^2} \, dy = [e^{y^2}]_0^2 = e^4 - 1$.)
**Ex 5** Determine the volume of the tetrahedron bounded by the coordinate planes and the plane $3x + 6y + 4z = 12 = 0$.

Let $\mathcal{R}$ be the triangular region in the $xy$-plane under the base of this tetrahedron.

Setting $z = 0$ in the equation gives $3x + 6y = 12$ or $y = 2 - \frac{x}{2}$ for this boundary curve.

$V(\text{tetrahedron}) = \iiint_{\mathcal{R}} \left(3 - \frac{3}{4}x - \frac{3}{2}y\right) \, dx \, dy \, dz$

\[ = \int_0^1 \left( \int_0^{2-\frac{x}{2}} \left(3 - \frac{3}{4}x - \frac{3}{2}y\right) \, dy \right) \, dx \]

\[ = \int_0^1 \left[ 3y - \frac{3}{4}xy - \frac{3}{2}y^2 \right]_{y=0}^{y=2-\frac{x}{2}} \, dx \]

\[ = \ldots = 4. \]

---

**Ex 4** Find $\int_0^4 \left( \int_{x/2}^{2y} e^{y^2} \, dy \right) \, dx$.

Well, of course you can't differentiate $e^{y^2}$. But you also can't just "exchange the integrals" — you'd end up with oor limits that aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration:

The integral above is

\[ \iiint_{\mathcal{R}} e^{y^2} \, dA = \int_0^2 \left( \int_0^{2y} e^{y^2} \, dx \right) \, dy \]

(b/c $\mathcal{R} = \{(x,y) \mid 0 \leq x \leq 4, \frac{x}{2} \leq y \leq 2\}$

\[ = \{(x,y) \mid 0 \leq y \leq 2, 0 \leq x \leq 2y\} \]

\[ = \int_0^2 2ye^{y^2} \, dy = \left[ e^{y^3} \right]_0^2 = e^4 - 1. \]
Some theory

Let $Q = [a, b] \times [c, d]$ be a closed rectangle, $D \subseteq Q$ a subset.
Some theory

Let $Q = [a,b] \times [c,d]$ be a closed rectangle, $D \subseteq Q$ a subset.

Definition: $D$ has content zero if for each $\varepsilon > 0$, there exists a finite union of rectangles of total area $\leq \varepsilon$ and containing $D$. 
Some theory

Let $Q = [a,b] \times [c,d]$ be a closed rectangle, $D \subseteq Q$ a subset.

**Definition:** $D$ has *content zero* if for each $\varepsilon > 0$, there exists a finite union of rectangles of total area $< \varepsilon$ and containing $D$.

**Theorem:** Let $F: Q \to \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

- continuous functions on $Q$
- step functions on $Q$ (why?)
- what else?
Some theory

Let $Q = [a, b] \times [c, d]$ be a closed rectangle, $D \subseteq Q$ a subset.

**Definition**: $D$ has **content zero** if for each $\varepsilon > 0$, there exists a finite union of rectangles of total area $< \varepsilon$ and containing $D$.

**Theorem**: Let $F : Q \rightarrow \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

**Proposition**: Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.
Some theory

Let $Q = [a,b] \times [c,d]$ be a closed rectangle, $D \subseteq Q$ a subset.

Definition: $D$ has content zero if for each $\epsilon > 0$, there exists a finite union of rectangles of total area $< \epsilon$ and containing $D$.

Theorem: Let $F : Q \to \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

Proposition: Points, line segments, graphs of continuous functions (and finite unions of these) have content zero. Apply the Theorem to $f = \frac{1}{10} \chi_Q$

Corollary: Continuous functions on $S$ of type I\&II (w/piecewise cts. boundary curves) are integrable.
Some theory

Let $Q = [a,b] \times [c, d]$ be a closed rectangle, $D \subseteq Q$ a subset.

Definition: $D$ has content zero if for each $\varepsilon > 0$, there exists a finite union of rectangles of total area $< \varepsilon$ and containing $D$.

Theorem: Let $F: Q \to \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

Proposition: Points, line segments, graphs of continuous functions (and finite unions of those) have content zero.

Corollary: Continuous functions on $S$ of type I or II (w/piecewise cts. boundary curves) are integrable.

Proof of Theorem:

STEP 1: Continuous functions on $Q$

Lecture 33 Theorem A $\Rightarrow F$ bounded.

So by Lecture 37, $I \& I'$ exist.
Some theory

Let \( Q = [a,b] \times [c,d] \) be a closed rectangle, \( D \subseteq Q \) a subset.

**Definition:** \( D \) has **content zero** if for each \( \varepsilon > 0 \), there exists a finite union of rectangles of total area \( < \varepsilon \) and containing \( D \).

**Theorem:** Let \( F: Q \to \mathbb{R} \) be continuous on \( Q \setminus D \), with \( D \) of content zero. Then \( F \) is integrable.

**Proposition:** Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

**Corollary:** Continuous functions on \( S \) of type I, II, III (or piecewise cts., boundary curves) are integrable.

---

Proof of Theorem:

**STEP 1:** Continuous functions on \( Q \)

Lecture 33 Theorem A \( \Rightarrow \) \( F \) bounded.
So by Lecture 37, \( I \) & \( \overline{I} \) exist.
Lecture 33 Theorem C \( \Rightarrow \) \( F \) uniformly cts.

\[ \Rightarrow \] for each fixed \( \varepsilon > 0 \),

\( F \) partition \( P \) of \( Q \) s.t. on each \( Q_{ij} \)
the difference of the maximum \( M_{ij}(F) \)
and the minimum \( m_{ij}(F) \) is \( < \varepsilon \).
Some theory

Let $Q = [a,b] \times [c,d]$ be a closed rectangle, $D \subseteq Q$ a subset.

**Definition:** $D$ has content zero if for each $\varepsilon > 0$, there exists a finite union of rectangles of total area $< \varepsilon$ and containing $D$.

**Theorem:** Let $F : Q \to \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

**Proposition:** Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

**Corollary:** Continuous functions on $S$ of type $I$ and $II$ (i.e., piecewise cts. boundary curves) are integrable.

Proof of Theorem:

**STEP 1:** Continuity functions on $Q$

Lecture 33 Theorem A $\Rightarrow$ $F$ bounded.

So by Lecture 37, $I$ and $\bar{I}$ exist.

Lecture 33 Theorem C $\Rightarrow$ $F$ uniformly cts.

$\Rightarrow$ for each fixed $\varepsilon > 0$, $F$ partitions $P$ of $Q$ s.t. on each $Q_{ij}$ the difference of the maximum $M_{ij}(F)$ and the minimum $m_{ij}(F)$ is $< \varepsilon$.

Define $D_{ij} \subseteq Q_{ij} \Rightarrow m_{ij}(F)$ and $\bar{D}_{ij} \subseteq Q_{ij} \Rightarrow M_{ij}(F)$ so that $D \leq F \leq \bar{D}$.
Some theory

Let \( Q = [a,b] \times [c,d] \) be a closed rectangle, \( D \subseteq Q \) a subset.

Definition: \( D \) has **content zero** if for each \( \varepsilon > 0 \), there exists a finite union of rectangles of total area \( \varepsilon \) and containing \( D \).

**Theorem:** Let \( F: Q \to \mathbb{R} \) be continuous on \( Q \setminus D \), with \( D \) of content zero. Then \( F \) is integrable.

Proposition: Points, line segments, graphs of continuous functions (and finite unions thereof) have content zero. Apply the theorem to \( f = \int_0^{\alpha} g(x) \).

Corollary: Continuous functions on \( g \) of type \( I \& II \) \( ( \text{w/piecewise cts. boundary curves}) \) are integrable.

Proof of Theorem:

**STEP 1:** Continuous functions on \( Q \)

Lecture 33 Theorem A \( \Rightarrow F \) bounded.

So by Lecture 37, \( I \) \& \( \bar{I} \) exist.

Lecture 33 Theorem C \( \Rightarrow F \) uniformly cts.

\( \Rightarrow \) for each fixed \( \varepsilon > 0 \), \( F \) partitions \( P \) of \( Q \) s.t. on each \( Q_{ij} \)
the difference of the maximum \( M_{ij}(F) \) and the minimum \( m_{ij}(F) \) is \( \leq \varepsilon \).

Define \( \delta_i, \theta_j : = m_{ij}(F) \) and \( \Delta_i, \Xi_j : = M_{ij}(F) \)
so that \( \delta \leq F \leq \Xi \). Then

\[
\sum m_{ij}(F) a(Q_{ij}) = \iint_Q \delta \, dA \leq I \\
\leq \bar{I} \leq \iint_Q \Xi \, dA = \sum M_{ij}(F) a(Q_{ij})
\]

while the difference of the RHS \(-\) LHS \( \leq \varepsilon \cdot a(Q) \).
Some theory

Let \( Q = [a,b] \times [c, d] \) be a closed rectangle, \( D \subseteq Q \) a subset.

Definition: \( D \) has content zero if for each \( \varepsilon > 0 \), there exists a finite union of rectangles of total area \( \varepsilon \) and containing \( D \).

Theorem: Let \( F : Q \to \mathbb{R} \) be continuous on \( Q \setminus D \), with \( D \) of content zero. Then \( F \) is integrable.

Proposition: Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

Corollary: Continuous functions on \( Q \) of type \( I^I \) (or piecewise cts. boundary curve) are integrable.

Proof of Theorem:

**STEP 1**: Continuous functions on \( Q \)

Lecture 33 Theorem A \( \Rightarrow \) \( F \) bounded.

So by Lecture 37, \( I \) and \( \bar{I} \) exist.

Lecture 33 Theorem C \( \Rightarrow \) \( F \) uniformly cts.

\( \Rightarrow \) for each fixed \( \varepsilon > 0 \),

if partition \( P \) of \( Q \) s.t. on each \( Q_{ij} \)

the difference of the maximum \( M_{ij}(F) \)

and the minimum \( m_{ij}(F) \) is \( < \varepsilon \).

Define \( d/\varepsilon \), \( i = m_{ij}(F) \) and \( t/\varepsilon \), \( j = M_{ij}(F) \)

so that \( d \leq F \leq t \). Then

\[ \sum_{ij} m_{ij}(F) a(Q_{ij}) = \int_Q d \, dA \leq I \]

\[ \leq \bar{I} \leq \int_Q t \, dA = \sum_{ij} M_{ij}(F) a(Q_{ij}) \]

while the difference of the RHS - LHS \( < \varepsilon \cdot a(Q) \).

So \( 0 \leq \bar{I} - \overline{I} < \varepsilon \cdot a(Q) \), and since \( \varepsilon > 0 \)

was arbitrary, \( \overline{I} = \bar{I} \) and \( F \) is integrable.
Some theory

Let \( Q = [a,b] \times [c,d] \) be a closed rectangle, \( D \subseteq Q \) a subset.

**Definition:** \( D \) has content zero if for each \( \varepsilon > 0 \), there exists a finite union of rectangles of total area \( \leq \varepsilon \) and containing \( D \).

**Theorem:** Let \( F: Q \to \mathbb{R} \) be continuous on \( Q \setminus D \), with \( D \) of content zero. Then \( F \) is integrable.

**Proposition:** Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

**Corollary:** (Continuous functions on \( S \) of type \( I \) or \( II \) w/piecewise cts. boundary curves) are integrable.

---

**Proof of Theorem:**

**STEP 2:** The general case

Let \( \delta > 0 \) be given, and choose a partition \( P \) of \( Q \) so that amongst the \( Q_{ij} \) are some rectangles as described in the definition.
Some theory

Let \( Q = [a,b] \times [c,d] \) be a closed rectangle, \( D \subseteq Q \) a subset.

Definition: \( D \) has content zero if for each \( \varepsilon > 0 \), there exists a finite union of rectangles of total area \( < \varepsilon \) and containing \( D \).

Theorem: Let \( F: Q \to \mathbb{R} \) be continuous on \( Q \setminus D \), with \( D \) of content zero. Then \( F \) is integrable.

Proposition: Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

Corollary: Continuous functions on \( Q \) of type \( I/II \) (or piecewise cts. boundary curves) are integrable.

Proof of Theorem:

**STEP 2: The general case**

Let \( \delta > 0 \) be given, and choose a partition \( P \) of \( Q \) so that amongst the \( Q_{ij} \) are some rectangles as described in the Definition.

Define step functions \( s \) & \( t \) as before; except on the \( Q_{ij} \) covering \( D \) we set \( s = -M \) and \( t = M \) (where \( |f| \leq M \) on \( Q \)).
Some theory

Let \( Q = [a, b] \times [c, d] \) be a closed rectangle, \( D \subseteq Q \) a subset.

Definition: \( D \) has content zero if for each \( \varepsilon > 0 \), there exists a finite union of rectangles of total area \( < \varepsilon \) and containing \( D \).

Proposition: Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

Corollary: Continuous functions on \( Q \) of type \( I/Q \) \( \circ \) (piecewise cts. boundary curves) are integrable.

Proof of Theorem:

**STEP 2: The general case**

Let \( \delta > 0 \) be given, and choose a partition \( P \) of \( Q \) so that amongst the \( Q_{ij} \) are some rectangles as described in the Definition.

Define step functions \( s, t \) as before; except on the \( Q_{ij} \) covering \( D \), we set \( s = -M \) and \( t = M \) (when \( |F(x)| \leq M \) on \( Q \)).

Arguing as in Step 1, we get
\[
0 \leq \overline{I} - \underline{I} < 2M \delta + \varepsilon \cdot a(Q)
\]

Letting \( \varepsilon, \delta \to 0 \), we get
\[
\overline{I} = \underline{I} = 0
\]
Some theory

Let $Q = [a, b] \times [c, d]$ be a closed rectangle, $D \subseteq Q$ a subset.

**Definition:** $D$ has content zero if for each $\varepsilon > 0$, there exists a finite union of rectangles of total area $\leq \varepsilon$ and containing $D$.

**Theorem:** Let $F : Q \to \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

**Proposition:** Points, line segments, graphs of continuous functions (and finite unions of these) have content zero. Apply the Theorem to $f = \int 0 \, Q\setminus D$.

**Corollary:** Continuous functions on $Q$ of type $I\&II$ (w/piecewise cts. boundary curves) are integrable.

Proof of Proposition:

We just need to check that the graph $\Gamma$ of a continuous function $g : [a, b] \to \mathbb{R}$ has content zero.
Some theory

Let $Q = [a,b] \times [c,d]$ be a closed rectangle, $D \subseteq Q$ a subset.

**Definition:** $D$ has *content zero* if for each $\varepsilon > 0$, there exists a finite union of rectangles of total area $< \varepsilon$ and containing $D$.

**Theorem:** Let $F : Q \to \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

**Proposition:** Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

**Corollary:** Continuous functions on $S$ of type I&II (w/piecewise cts. boundary curves) are integrable.

---

Proof of Proposition:

We just need to check that the graph $\Gamma$ of a continuous function $g : [a,b] \to \mathbb{R}$ has content zero. Fix $\varepsilon > 0$.

Since $g$ is uniformly continuous, we can partition $[a,b]$ into finitely many intervals $[x_{i-1}, x_i] = d_i$ s.t. the span (= max-min) of $g$ on each $d_i$ is $< \frac{\varepsilon}{b-a}$.
Some theory

Let $Q = [a, b] \times [c, d]$ be a closed rectangle, $D \subseteq Q$ a subset.

Definition: $D$ has content zero if for each $\epsilon > 0$, there exists a finite union of rectangles of total area $< \epsilon$ and containing $D$.

Theorem: Let $F: Q \rightarrow \mathbb{R}$ be continuous on $Q \setminus D$, with $D$ of content zero. Then $F$ is integrable.

Proposition: Points, line segments, graphs of continuous functions (and finite unions of these) have content zero.

Corollary: Continuous functions on $[a,b]$ of type $I$ or $II$ (continuous or piecewise continuous, respectively) are integrable.

Proof of Proposition:

We just need to check that the graph $\Gamma$ of a continuous function $g: [a, b] \rightarrow \mathbb{R}$ has content zero. Fix $\epsilon > 0$.

Since $g$ is uniformly continuous, we can partition $[a, b]$ into finitely many intervals $[x_{i-1}, x_i] = I_i$ s.t. the span of $g$ on each $I_i$ is $< \frac{\epsilon}{b-a}$.

The sum of the areas of the boxes in the picture is therefore $< \epsilon$. \qed