Lecture 39 : Theory of Double Integrals

Rectangular case

Let $Q = [a,b] \times [c,d]$ be a closed rectangle, $f : Q \to \mathbb{R}$ a function we'd like to integrate using iterated integrals. In order for Fubini to apply, we need to know that f is integrable (i.e. that $\iint_Q f \, dA$ exists in the first place).

<u>Theorem 1</u>: If f is continuous (on Q), then it is integrable (on Q).

<u>Proof</u>: • Lect. 33 Thm. A $\Rightarrow$ f is bounded $\underset{\text{Lecture 38}}{\Longrightarrow}$ $\overline{I}$ & $\underline{I}$ exist.

• Lect. 33 Thm. C $\Rightarrow$ f is <u>uniformly</u> continuous $\Rightarrow$

for each fixed $\epsilon > 0$, $\exists$ partition P of Q s.t. on each Q_{ij} the difference of the maximum $M_{ij}(f)$ & minimum $m_{ij}(f)$ is $< \epsilon$.

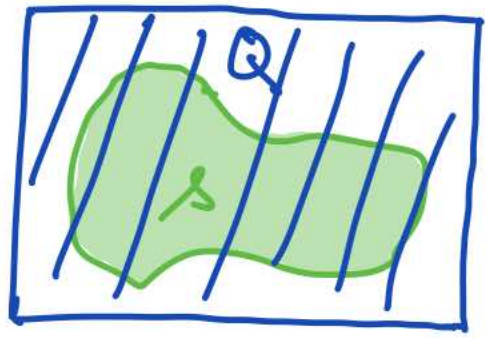
• Let s, t be step functions with $s|_{Q_{ij}^\circ} := m_{ij}(f)$, $t|_{Q_{ij}^\circ} := M_{ij}(f)$ (so that $s \le F \le t$). Then

$$\underbrace{\sum_{i,j} m_{ij}(f)\, a(Q_{ij})} = \iint_Q s \, dA \le \underline{I} \le \overline{I} \le \iint_Q t \, dA = \underbrace{\sum_{i,j} M_{ij}(f)\, a(Q_{ij})}$$

while the difference of the RHS $-$ LHS $< \epsilon \cdot a(Q)$.

• So $0 \le \overline{I} - \underline{I} < \epsilon \cdot a(Q)$, and as $\epsilon > 0$ was arbitrary, $\overline{I} = \underline{I}$ (i.e. f is integrable). $\square$

Non-rectangular case



Let S be a closed, bounded subset of $\mathbb{R}^2$, and enclose it in a closed rectangle Q. If $f(x,y)$ is a function on S, define a function on Q by

$$\tilde{f}(x,y) := \begin{cases} f(x,y), & (x,y) \in S \\ 0, & (x,y) \in Q \setminus S. \end{cases}$$

Definition: f is integrable (on S) $\iff$ $\tilde{f}$ is integrable (on Q). In this case, $\iint_S f \, dA := \iint_Q \tilde{f} \, dA$.

Theorem 2: Let $g : Q \to \mathbb{R}$ be a bounded function which is continuous on $Q \setminus D$, where $D \subset Q$ is a subset of content zero. (That is, for each $\epsilon > 0$ there exists a finite union of rectangles of total area $< \epsilon$ and containing D.) Then g is integrable.

Proof: Let $\delta > 0$ be given, & choose a partition P of Q so that amongst the $\{Q_{ij}\}$ are some rectangles as described in the parenthetical. Define step functions s & t as before; except on the $\{Q_{ij}\}$ covering D we set $s = -M$ and $t = M$ (where $|g| \leq M$ on Q). Arguing as in the last proof, we get

$$0 \leq \overline{I} - \underline{I} < 2M\delta + \epsilon \cdot a(Q).$$

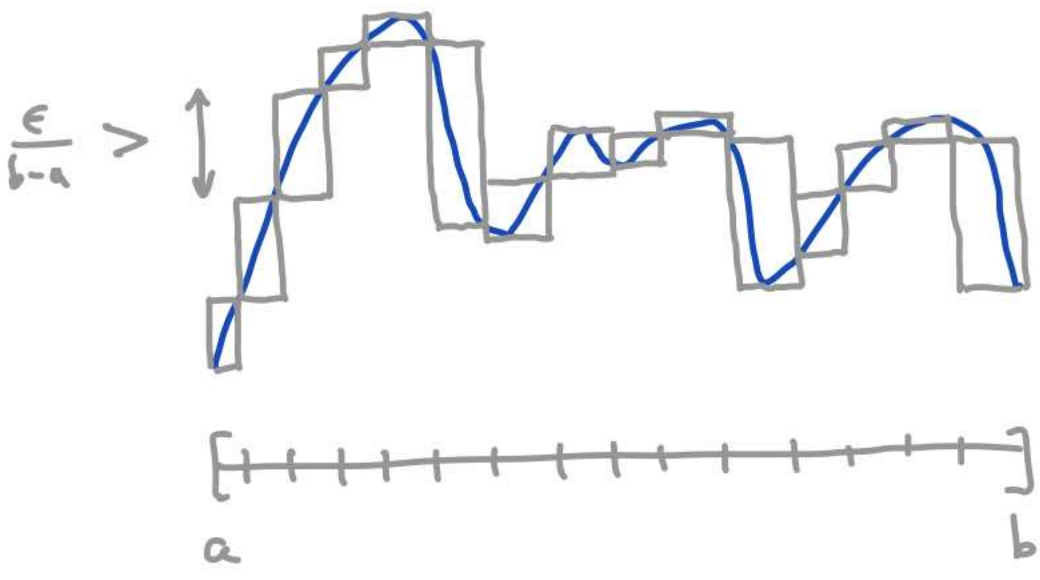
Letting $\epsilon, \delta \to 0$ forces $\overline{I} = \underline{I}$. $\square$

__Proposition__: Points, line segments, graphs of continuous functions, and finite unions of these have content zero.

[So basically, if the boundary of S is nice enough, you can take $g = \tilde{f}$ in Theorem 2. (Note that Thm. 2 also includes the integrability of step functions, though of course we already knew that.)]

__Proof__: We just need to check that the graph Γ of a continuous function $\varphi : [a,b] \to \mathbb{R}$ has content zero. Fix $\epsilon > 0$.

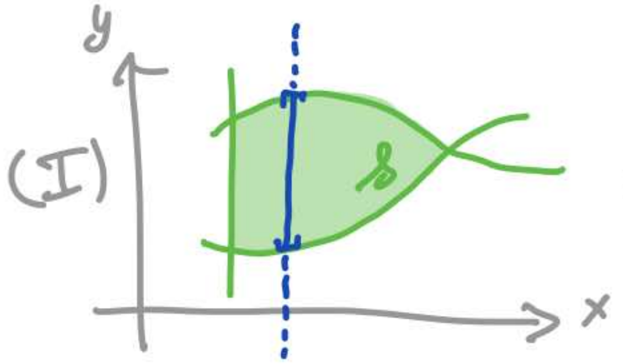
Since φ is uniformly continuous, we can partition $[a,b]$ into finitely many intervals $[x_{i-1}, x_i] = J_i$ s.t. the span (= max − min) of φ on each J_i is $< \frac{\epsilon}{b-a}$.

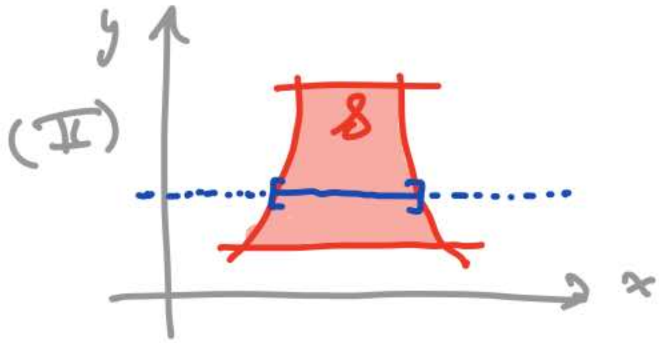


The sum of the areas of the boxes in the picture is therefore $< \epsilon$. $\square$

Computing non-rectangular double integrals

<u>Definition</u>: S is of type $\begin{cases} \mathrm{I} \\ \mathrm{II} \end{cases}$ if each line parallel to the $\begin{cases} y \\ x \end{cases}$-axis intersects S in a single closed interval (or a point, or not at all).



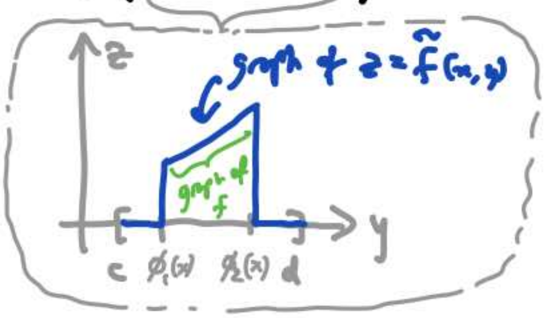


So a type I set takes the form

$$S = \{ (x,y) \mid a \le x \le b \ , \ \phi_1(x) \le y \le \phi_2(x) \}$$

for some functions $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$. Enclosing it in $Q = [a,b] \times [c,d]$,

$$\iint_S f \, dA = \iint_Q \tilde{f} \, dA \underset{\text{Fubini}}{=} \int_a^b \underbrace{\left(\int_c^d \tilde{f}(x,y) \, dy \right)}_{} dx = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y) \, dy \right) dx$$



Similarly, a type II set may be written

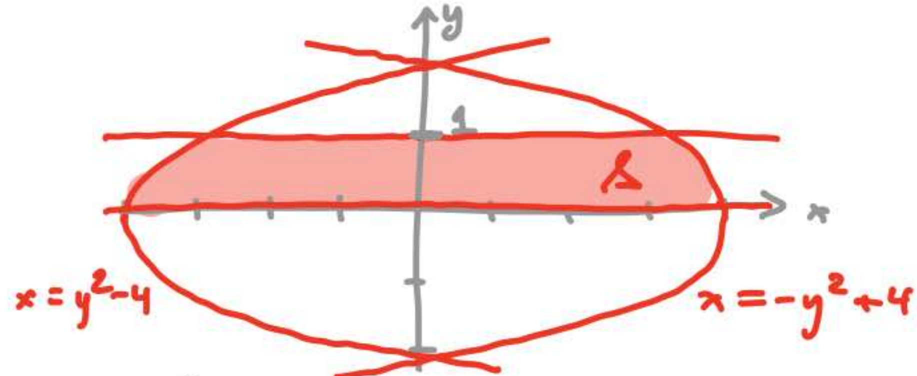
$$S = \{ (x,y) \mid c \le y \le d \ , \ \psi_1(y) \le x \le \psi_2(y) \},$$

and

$$\iint_S f \, dA = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y) \, dx \right) dy.$$

Both of these cases are covered by Theorem 2.

Ex 1 / Compute $\iint_{\mathcal{S}} (y+1)\, dA$



$\mathcal{S}$ is both of type I & type II, but the functions for the type II integration are easier (and already written):

$$\iint_{\mathcal{S}} (y+1)\,dA = \int_0^1 \left(\int_{y^2-4}^{-y^2+4} \underbrace{(y+1)}_{\text{constant wrt } x}\, dx \right) dy = \int_0^1 \Big[x(y+1) \Big]_{x=y^2-4}^{-y^2+4} dy$$

$$= \int_0^1 \left(-y^2+4-(y^2-4)\right)(y+1)\, dy = \cdots = \frac{65}{6}$$

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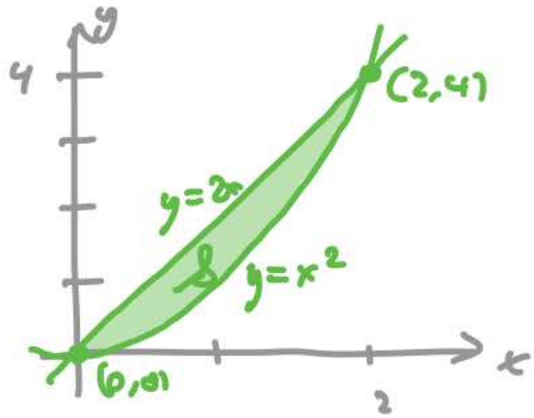
Ex 2 / Find $\iint_{\mathcal{S}} (8x+10y)\, dA$, where $\mathcal{S}$ is the region between the graphs of $y=x^2$ and $y=2x$.



$$\iint_{\mathcal{S}} \cdots = \int_0^2 \left(\int_{x^2}^{2x} (8x+10y)\, dy \right) dx$$

$$= \int_0^2 \Big[8xy + 5y^2 \Big]_{y=x^2}^{2x} dx$$

$$= \int_0^2 \left((16x^2+20x^2) - (8x^3+5x^4) \right) dx = \int_0^2 \left(-5x^4 - 8x^3 + 36x^2\right) dx$$

$$= \cdots = 32.$$

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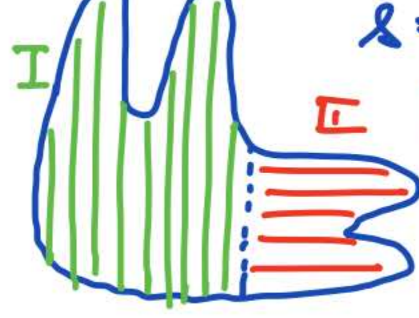
What if $\mathcal{S}$ is more complicated?

Split it into subsets and write

$$\iint_{\mathcal{S}} = \iint_{\mathcal{S}_I} + \iint_{\mathcal{S}_{II}}$$



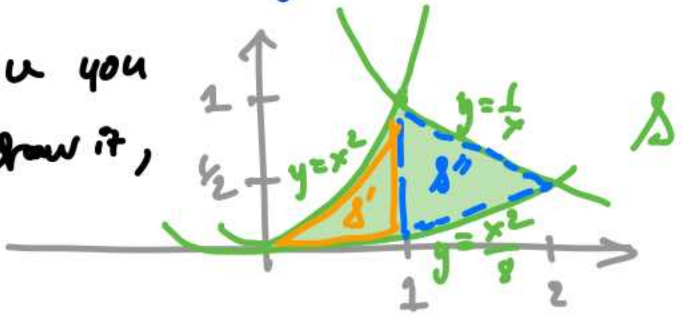
$\mathcal{S} = \mathcal{S}_I \cup \mathcal{S}_{II}$ (and $\mathcal{S}_I \cap \mathcal{S}_{II}$ is a curve)

More generally, you can use this technique (breaking the region) to compute integrals which are of type I or type II:

i.e. area of S

Ex 3/ Find $\iint_S 1\,dA$, where S is in the 1^{st} quadrant, bounded by $y = x^2$, $y = \frac{x^2}{8}$, and $y = \frac{1}{x}$.

Once you draw it,



you see the need for dividing it in two:

$$\iint_S 1\,dA = \iint_{S'} 1\,dA + \iint_{S''} 1\,dA$$

$$= \int_0^1 \int_{x^2/8}^{x^2} 1\,dy\,dx + \int_1^2 \int_{x^2/8}^{1/x} 1\,dy\,dx = \int_0^1 \left(x^2 - \frac{x^2}{8}\right)dx + \int_1^2 \left(\frac{1}{x} - \frac{x^2}{8}\right)dx$$

$$= \ldots = \log(2).$$

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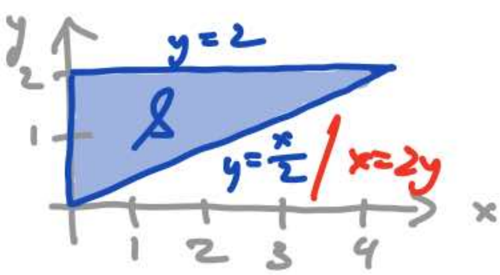
Ex 4/ Find $\int_0^4 \left(\int_{x/2}^2 e^{y^2}\,dy\right)dx$.

Well, of course you can't antidifferentiate e^{y^2}.

But you also can't just "exchange the integrals" — you'd end up with outer limits which aren't constants, which makes no sense. Instead, draw the region to figure out how to correctly switch the variables of integration. The integral above



$$= \iint_S e^{y^2}\,dA = \int_0^2 \left(\int_0^{2y} e^{y^2}\,dx\right)dy$$

(b/c $S = \{(x,y) \mid 0 \le x \le 4,\ \frac{x}{2} \le y \le 2\}$
$= \{(x,y) \mid 0 \le y \le 2,\ 0 \le x \le 2y\}$)

$$= \int_0^2 2y e^{y^2}\,dy = \left[e^{y^2}\right]_0^2 = e^4 - 1.$$

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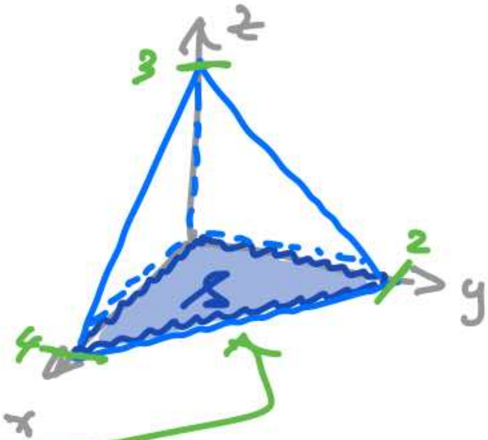
One more example ...

Ex 5 / Determine the volume of the tetrahedron bounded by the coordinate planes and the plane $3x+6y+4z-12=0$.

$$\hookrightarrow z = 3 - \frac{3}{4}x - \frac{3}{2}y$$

Let S be the triangular region in the xy-plane under the base of this tetrahedron. Setting $z=0$ in the equation gives $3x+6y=12$ or $y = 2 - \frac{x}{2}$ for this boundary curve.



$$\begin{aligned}
V(\text{tetrahedron}) &= \iint_S \left(3 - \frac{3}{4}x - \frac{3}{2}y\right) dA \\
&= \int_0^4 \int_0^{2-\frac{x}{2}} \left(3 - \frac{3}{4}x - \frac{3}{2}y\right) dy\, dx \\
&= \int_0^4 \left[3y - \frac{3}{4}xy - \frac{3}{4}y^2\right]_{y=0}^{2-\frac{x}{2}} dx \\
&= \cdots = 4.
\end{aligned}$$

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