Lecture 39

Change of variable in the $SS_{DA}$
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... but first, some applications and examples...
Mass

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If a lamina covers a region \( \mathcal{S} \subset \mathbb{R}^2 \)

with mass density function \( \delta: \mathcal{S} \to \mathbb{R} \), we may extend \( \delta \) to \( \mathcal{Q} \) by defining it to be zero off \( \mathcal{S} \).
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with mass density function $\delta: \mathcal{S} \to \mathbb{R}$, we may extend $\delta$ to $\mathcal{Q}$ by defining it to be zero off $\mathcal{S}$. 

Pick representative points $q_{ij} = (x_{ij}, y_{ij}) \in \mathcal{Q}_{ij}$.

Then the mass $m(q_{ij}) = \delta(q_{ij}) a(\mathcal{Q}_{ij})$

$\Rightarrow$ total mass $m(\mathcal{S}) = m(\mathcal{Q}) = \sum_{i,j} \delta(q_{ij}) a(\mathcal{Q}_{ij})$. 

Mass

Lamina := flat sheet so thin we can think of it as 2-D; made of nonhomogeneous material (density can vary)

If a lamina covers a region \( \mathcal{B} \subset \mathbb{R}^2 \)

with mass density function \( \delta : \mathcal{B} \to \mathbb{R} \), we may extend \( \delta \) to \( \mathcal{B} \) by defining it to be zero off \( \mathcal{B} \).

Pick representative points \( q_{ij} = (x_{ij}, y_{ij}) \in Q_{ij} \).

Then the mass \( m(Q_{ij}) \approx \delta(q_{ij}) a(Q_{ij}) \)

\( \Rightarrow \) total mass \( m(\mathcal{B}) = m(Q) \approx \sum_{i,j} \delta(q_{ij}) a(Q_{ij}) \)

By the small-portion theorem, taking the size of the partition to 0 makes the integrals of the upper & lower step functions coincide, so

\[
  m = \int_{\mathcal{B}} \delta \, dA.
\]

(mass of lamina)
Pick representative points \( q_{ij} = (x_{ij}, y_{ij}) \in Q_{ij}. \)

Then the mass \( m(Q_{ij}) = \delta(q_{ij}) a(Q_{ij}) \)

\[ \implies \text{total mass} \quad m(Q) = \sum_{i,j} \delta(q_{ij}) a(Q_{ij}). \]

By the small-span theorem, taking the size of the partition to 0 makes the integrals of the upper and lower step functions coincide, so

\[ m = \iiint \delta \, dA. \quad \text{(mass of lamina)} \]

Now let's think about seesaws:

For this to balance, we need

\[ m_1 \left( \bar{x} - x_1 \right) = m_2 \left( x_2 - \bar{x} \right). \]

Equality of moments
Solving for $\bar{x}$ gives $(m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2$

$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$.

Pick representative points $q_{ij} = (x_{ij}, y_{ij}) \in Q_{ij}$.

Then the mass $m(Q_{ij}) = \delta(q_{ij}) a(Q_{ij})$.

$\Rightarrow$ total mass $m(\delta) = m(Q) = \sum_{i,j} \delta(q_{ij}) a(Q_{ij})$.

By the small-span theorem, taking the size of the partition to 0 makes the integrals of the upper and lower step functions coincide, so

$$m = \iiint_\delta \delta \, dA.$$  \[ \text{(mass of lamina)} \]

Now let's think about scenarios:

For this to balance, we need

$$m_1 (\bar{x} - x_1) = m_2 (x_2 - \bar{x}).$$
Solving for $\bar{x}$ gives $(m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2$.

$$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$  

For $n$ masses, the balance point would be

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i},"\infty(= \frac{\sum m_i y_i}{\sum m_i}).$$

Pick representative points $q_{ij} = (x_{ij}, y_{ij}) \in Q_{ij}$. Thus the mass $m(Q_{ij}) = \delta(q_{ij}) a(Q_{ij})$.

$$\Rightarrow \text{total mass } m(\mathcal{D}) = m(Q) = \sum_{i,j} \delta(q_{ij}) a(Q_{ij}).$$

By the small-span theorem, taking the size of the partition to 0 makes the integrals of the upper and lower step functions coincide, so

$$m = \iint_{\mathcal{D}} \delta \ dA \quad \text{(mass of lamina)}.$$  

Now let’s think about scales:

for this to balance, we need

$$m_1 (\bar{x} - x_1) = m_2 (x_2 - \bar{x}).$$
Solving for $\bar{x}$ gives $(m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2$

$\Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$.

For n masses, the balance point would be

$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}$

$\bar{y} = \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i}$.

Thinking of the lamina as a bunch of small masses (at $q_{ij}$) yields

$\bar{x} \approx \frac{\sum_{i,j} x_{ij} \delta(q_{ij}) a(Q_{ij})}{\sum_{i,j} \delta(q_{ij}) a(Q_{ij})}$

etc.

Pick representative points $q_{ij} = (x_{ij}, y_{ij}) \in Q_{ij}$. Then the mass $m(Q_{ij}) = \delta(q_{ij}) a(Q_{ij})$

$k \to$ total mass $m(S) = m(Q) = \sum_{i,j} \delta(q_{ij}) a(Q_{ij})$.

By the small - span theorem, taking the size of the partition to 0 makes the integrals of the upper & lower step functions coincide, so

$m = \iint_S \delta \, dA$.

(mass of lamina)

Now let's think about scales:

for this to balance, we need

$m_1 (\bar{x} - x_1) = m_2 (x_2 - \bar{x})$. 
Solving for \( \bar{x} \) gives \((m_1+m_2)\bar{x} = m_1x_1 + m_2x_2 \Rightarrow \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \).

For \( n \) masses, the balance point would be
\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \quad \text{(in 2-D, \( \bar{y} = \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i} \)).}
\]

Thinking of the lamina as a bunch of small masses (at \( q_{ij} \)) yields
\[
\bar{x} = \frac{\sum_{j} x_{ij} a(q_{ij})}{\sum_{j} a(q_{ij})} \quad \text{etc.}
\]

Hence (taking the limit)
\[
\bar{x} = \frac{\int_a x \delta_{x,y} \, da}{\int_a \delta_{x,y} \, da}, \quad \bar{y} = \frac{\int_a y \delta_{x,y} \, da}{\int_a \delta_{x,y} \, da}
\]

where \( M_x \) and \( M_y \) are the moments about the \( x \)-\( d \) \( y \)-axis, and \((\bar{x}, \bar{y})\) is the center of mass.
Solving for \( \bar{x} \) gives \((m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2 \)
\[ \Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \]

For \( n \) masses, the balance point would be
\[ \bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}. \]

Thinking of the lamina as a bunch of small masses \((at \ q_{ij})\) yields
\[ \bar{x} = \frac{\sum_{j} x_{ij} \delta_{Q_{ij}} a(Q_{ij})}{\sum_{j} \delta_{Q_{ij}} a(Q_{ij})}. \]

Hence (taking the limit)
\[ \bar{x} = \frac{\int_{A} x \delta_{\lambda} \, dA}{\int_{A} \delta_{\lambda} \, dA}, \quad \bar{y} = \frac{\int_{A} y \delta_{\lambda} \, dA}{\int_{A} \delta_{\lambda} \, dA}. \]

Where \( M_x \) and \( M_y \) are the moments about the \( x \) and \( y \)-axes, and \((\bar{x}, \bar{y})\) is the center of mass.

**Pappus's Theorem**

Suppose we now rotate \( A \) about the \( x \)-axis, so that the centroid trace is at a curve of radius \( \bar{y} \).
Solving for $\bar{x}$ gives $(m_1+m_2)\bar{x} = m_1x_1 + m_2x_2$

$\Rightarrow \bar{x} = \frac{m_1x_1 + m_2x_2}{m_1+m_2}$.

For n masses, the balance point would be

$\bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}$

$\bar{y} = \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i}$

Thinking of the lamina as a bunch of small masses (at $q_{ij}$) yields

$\bar{x} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} x_{ij} \delta(x_{ij}, y_{ij})}{\sum_{i=1}^{n} \sum_{j=1}^{m} \delta(x_{ij}, y_{ij})}$

$\bar{y} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} y_{ij} \delta(x_{ij}, y_{ij})}{\sum_{i=1}^{n} \sum_{j=1}^{m} \delta(x_{ij}, y_{ij})}$

hence (taking the limit)

$\bar{x} = \frac{\int_a^b x \delta(x,y) \, dy}{\int_a^b \delta(x,y) \, dy}$

$\bar{y} = \frac{\int_a^b y \delta(x,y) \, dy}{\int_a^b \delta(x,y) \, dy}$

where $M_x$ and $M_y$ are the moments about the $x$-d $y$-axes, and $(\bar{x}, \bar{y})$ is the center of mass.

**Pappus’s Theorem**

Suppose we now rotate $S$ about the $x$-axis, so that the centroid traces out a curve of radius $\bar{y}$. The resulting solid has volume

$V = \int_a^b \pi \left( \phi_2(x)^3 - \phi_1(x)^3 \right) \, dx$

by viewing $S$ as a type I region so that vertical cross-sections of the solid are annuli.
Solving for \( \bar{x} \) gives \( (m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2 \)
\[ \Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}. \]

For \( n \) masses, the balance point would be
\[ \bar{x} = \frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i} \]

(d is in 2-D, \( \bar{y} = \frac{\sum y_i m_i}{\sum m_i} \)).

Thinking of the lamina as a bunch of small masses (at \( q_{ij} \)) yields
\[ \bar{x} = \frac{\sum_{j} x_{ij} \delta(q_{ij}) a(Q_{ij})}{\sum_{j} \delta(q_{ij}) a(Q_{ij})} \]

hence (taking the limit)
\[ \bar{x} = \frac{\int_{S} x \delta(x, y) \, dA}{\int_{S} \delta(x, y) \, dA}, \quad \bar{y} = \frac{\int_{S} y \delta(x, y) \, dA}{\int_{S} \delta(x, y) \, dA}. \]

where \( M_x \) & \( M_y \) are the moments about the \( x \)-& \( y \)-axes, and \((\bar{x}, \bar{y})\) is the center of mass.

**Pappus's Theorem**

Suppose we now rotate \( S \) about the \( x \)-axis, so that the centroid traces out a curve of radius \( \bar{y} \). The resulting solid has volume
\[ V = \int_{a}^{b} \pi \left( \phi_2(x)^2 - \phi_1(x)^2 \right) \, dx \]
\[ = \pi \int_{a}^{b} \left( \int_{a}^{b} 2y \, dy \right) \, dx \]
\[ = 2\pi \int_{S} y \, dA \]
Solving for $\bar{x}$ gives $(m_1 + m_2) \bar{x} = m_1 x_1 + m_2 x_2 \Rightarrow \bar{x} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$.

For $n$ masses, the balance point would be

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i \cdot m_i}{\sum_{i=1}^{n} m_i}$$

Thinking of the lamina as a bunch of small masses $(at q_{ij})$ yields

$$\bar{x} \approx \frac{\sum_{j} x_{ij} \cdot \delta(q_{ij}) \cdot a(q_{ij})}{\sum_{j} \delta(q_{ij}) \cdot a(q_{ij})}$$

hence (taking the limit)

$$\bar{x} = \frac{\iint_{Q} x \cdot \delta(x,y) \, dA}{\iint_{Q} \delta(x,y) \, dA}, \quad \bar{y} = \frac{\iint_{Q} y \cdot \delta(x,y) \, dA}{\iint_{Q} \delta(x,y) \, dA}$$

where $M_x$ and $M_y$ are the moments about the $x$- and $y$-axes, and $(\bar{x}, \bar{y})$ is the center of mass.

**Pappus's Theorem**

Suppose we now rotate $Q$ about the $x$-axis, so that the centroid traces out a circle of radius $\bar{y}$. The resulting solid has volume

$$V = \int_{a}^{b} \pi \left( \phi_x^2 \cdot r^2 - \phi_x^3 \right) \, dx$$

$$= \pi \int_{a}^{b} \left( \int_{x}^{b} 2 y \, dy \right) \, dx$$

$$= 2\pi \int_{a}^{b} y \, dA = 2\pi M_x$$

$$= 2\pi \bar{y} \int_{a}^{b} dA = 2\pi \bar{y} a(B)$$

where the mass density $\delta = 1$. (In this case $(\bar{x}, \bar{y})$ is called the centroid of $B$.)
Pappus’s Theorem

Suppose we now rotate $S$ about the $x$-axis, so that the centroid traces out a circle of radius $\overline{y}$. The resulting solid has volume

$$V = \int_a^b \pi (y_2(x)^2 - y_1(x)^2) \, dx$$

$$= \pi \int_a^b \left( \int_{y_1(x)}^{y_2(x)} 2y \, dy \right) \, dx$$

$$= 2\pi \int_a^b y \, dx = 2\pi M_x$$

$$= 2\pi \overline{y} \int_S dA = 2\pi \overline{y} a(S)$$

where the mass density $\delta = 1$. (In this case $(\overline{x}, \overline{y})$ is called the centroid of $S$.) See Apostol for another theorem in this vein, concerning the centroid of a disjoint union $S_1 \cup S_2$. 


Ex 1. Find the volume of the torus $T$ depicted:

Pappus's Theorem

Suppose we now rotate $S$ about the $x$-axis, so that the centroid traces out a circle of radius $\overline{y}$. The resulting solid has volume

$$V = \int_a^b \pi \left( \phi(x)^2 - \phi_1(x)^2 \right) \, dx$$

$$= \pi \int_a^b \left( \int_a^b 2y \, dy \right) \, dx$$

$$= 2\pi \int_a^b y \, dA = 2\pi M_x$$

$$= 2\pi \overline{y} \int_S dA = 2\pi \overline{y} a(S)$$

where the mass density $\delta = 1$. (In this case $(\overline{x}, \overline{y})$ is called the centroid of $S$.) See Apostol for another theorem in this vein, concerning the centroid of a disjoint union $S_1 \cup S_2$. 
Ex 1 | Find the volume of the turn T depicted:
\[ v(T) = 2\pi \bar{y} a(S) \]
\[ = 2\pi R \cdot \pi r^2 = 2\pi^2 r^2 R \]

Pappus's Theorem

Suppose we now rotate \( S \) about the \( x \)-axis, so that the centroid traces out a circle of radius \( \bar{y} \). The resulting solid has volume
\[ V = \int_a^b \pi \left( \phi_x(x)^2 - \phi_y(x)^3 \right) \, dx \]
\[ = \pi \int_a^b \left( \int_0^{\phi_y(x)} 2y \, dy \right) \, dx \]
\[ = 2\pi \int_a^b y \, dA = 2\pi M_x \]
\[ = 2\pi \bar{y} \int_a^b dA = 2\pi \bar{y} a(S) \]

where the mass density \( \delta \equiv 1 \). (In this case \((\bar{x}, \bar{y})\) is called the centroid of \( S \).) See Apostol for another theorem in this vein, concerning the centroid of a disjoint union \( S_1 \cup S_2 \).
Ex 1 \text{ Find the volume of the torus } T \text{ depicted:}\\\v(T) = 2\pi \overline{y} a(8)\\ = 2\pi R \cdot \pi r^2 = 2\pi^2 r^2 R.\\\\Ex 2 \text{ Find the centroid of the quarter-circle shown:}\\\\Pappus's Theorem\\Suppose we now rotate } \mathcal{B} \text{ about the } x\text{-axis, so that the centroid traces out a circle of radius } \overline{y}. \text{ The resulting solid has volume}\\V = \int_a^b \pi ((\phi_2(x))^2 - \phi_1(x))^2) \, dx\\= \pi \int_a^b (\int_0^{\phi_2(x)} 2y \, dy) \, dx\\= 2\pi \int_a^b y \, dA = 2\pi M_x\\= 2\pi \overline{y} \int_a^b dA = 2\pi \overline{y} a(8)\\\text{where the mass density } \delta = 1. \text{ (In this case } (\overline{x}, \overline{y}) \text{ is called the centroid of } \mathcal{B}.) \text{ See Apostol for another theorem in this vein, concerning the centroid of a disjoint union } \mathcal{B}_1 \cup \mathcal{B}_2.
Ex 1 Find the volume of the torus $T$ depicted:

$$V(T) = 2\pi \bar{y} a(\theta)$$

$$= 2\pi R \cdot \pi r^2 = 2\pi^2 r^2 R.$$ 

Ex 2 Find the centroid of the quarter-circle shown:

Rotating about the $x$-axis yields a hemisphere of volume

$$\frac{2}{3} \pi r^3 = V = 2\pi \bar{y} a(\theta).$$

Pappus's Theorem

Suppose we now rotate $\delta$ about the $x$-axis, so that the centroid traces out a circle of radius $\bar{y}$. The resulting solid has volume

$$V = \int_a^b \pi \left( \phi_2(x)^2 - \phi_1(x)^2 \right) \, dx$$

$$= \pi \int_a^b \left( \int_0^{\phi_2(x)} 2y \, dy \right) \, dx$$

$$= 2\pi \int_a^b y \, dA = 2\pi \bar{y} a(\delta)$$

where the mass density $\delta = 1$. (In this case $(\bar{r}, \bar{y})$ is called the centroid of $\delta$.) See Apostol for another theorem in this vein, concerning the centroid of a disjoint union $\delta_1 \cup \delta_2$. 
Ex 1 Find the volume of the torus \( T \) depicted:
\[
v(T) = 2\pi \bar{y} a(b)
= 2\pi R \cdot \pi r^2
= 2\pi^2 r^2 R.
\]

Ex 2 Find the centroid of the quarter-circle shown:

Rotating about the \( x \)-axis yields a hemisphere of volume
\[
\frac{2}{3} \pi r^3 = V = 2\pi \bar{y} a(b) = \frac{\pi^2 r^2 \bar{y}}{2}
\]

where the mass density \( \sigma = 1 \). (In this case, \((\bar{x},\bar{y})\) is called the **centroid** of \( B \).) See Apostol for another theorem in this vein, concerning the centroid of a disjoint union \( B_1 \cup B_2 \).
Though I tend not to write it out explicitly, we have been doing a lot of substitutions in the integrals we compute. Recall how this goes: given

\[ u \mapsto g(u) = x \quad \text{a } C^1 \text{ function} \]

mapping

\[ [c, d] \mapsto [a, b] \]

in 1-to-1 fashion

\[ a = g(c), \ b = g(d) \]
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mapping
\[ [c, d] \longrightarrow [a, b] \quad \text{a} = g(c), \ b = g(d) \]
in 1-to-1 fashion,
\[ \int_a^b f(x) \, dx = \int_c^d f(g(u)) \, g'(u) \, du. \]

We are after a 2-variable analogue of this.
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We are after a 2-variable analogue of this.

Suppose
\[ \text{is a } 1\text{-}1 \land \text{onto } C^1 \text{ map.} \]
Though I tend not to write it out explicitly, we have been doing a lot of substitutions in the integrals we compute. Recall how this goes: given
\[ u \mapsto g(u) = x \]
a \( C^1 \) function
mapping
\[ [c,d] \mapsto [a,b] \]
\[ a = g(c), \quad b = g(d) \]
in 1-to-1 fashion,
\[ \int_a^b f(x) \, dx = \int_c^d f(g(u)) \, g'(u) \, du. \]
We are after a 2-variable analogue of this.

Suppose
\[ (u,v) \mapsto \vec{G}(u,v) = (x(u,v), y(u,v)) \]
and not worry about \( \vec{G} \) failing to be 1-1 on the boundary, or more generally a content-zero subset.
Though I tend not to write it out explicitly, we have been doing a lot of substitutions in the integrals we compute. Recall how this goes: given
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mapping
\[ [c, d] \mapsto [a, b] \]
\[ a = g(c), \quad b = g(d) \]
in 1-to-1 fashion,
\[ \int_{a}^{b} f(x) \, dx = \int_{c}^{d} f(g(u)) \cdot g'(u) \, du. \]
We are after a 2-variable analogue of this.

Suppose
\[ \tilde{G} \]  
is a 1-1 \( \text{onto} \) \( C^1 \) map.

We'll write
\[ (u,v) \mapsto \tilde{G}(u,v) = (x(u,v), y(u,v)) \]
and not worry about \( \tilde{G} \) failing to be 1-1 on the boundary, or more generally a content-zero subset. Then for any continuous function \( f : \mathcal{R} \to \mathbb{R} \),

\[ \iint_{\mathcal{R}} f \, dA = \iint_{\tilde{G}} f \circ \tilde{G} \cdot |J_{\tilde{G}}| \, dA \quad (\ast) \]
Though I tend not to write it out explicitly, we have been doing a lot of substitutions in the integrals we compute. Recall how this goes: given
\[ u \rightarrow g(u) = x \]
\( C^1 \) function mapping
\[ [c, d] \rightarrow [a, b] \]
\[ a = g(c), \quad b = g(d) \]
in 1-to-1 fashion,
\[ \int_a^b f(x) \, dx = \int_c^d f(g(u)) \, g'(u) \, du. \]
We are after a 2-variable analogue of this.

Suppose
\[ \text{ is a } 1-1 \text{ /onto } C^1 \text{ map.} \]

We'll write
\[ (u, v) \rightarrow \tilde{G}(u,v) = (x(u,v), y(u,v)), \]
and not worry about \( \tilde{G} \) failing to be 1-1 on the domain, or more generally a content-zero subset. Then for any continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \),

\[ \iint_S f \, dA = \iint_{\frac{1}{J} T} f \circ \tilde{G} \cdot |J_{\tilde{G}}| \, dA \]  \( \star \)

\[ \iint_S f(x,y) \, dx \, dy = \iint_T f(\tilde{G}(u,v)) \, det(J_{\tilde{G}}(u,v)) \, du \, dv \]

where
\[ J_{\tilde{G}} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \]

is the Jacobian or total derivative of \( \tilde{G} \).
We'll write
\[(u, v) \mapsto \tilde{G}(u, v) = (x(u, v), y(u, v)),\]
and not worry about \(\tilde{G}\) failing to be 1-1 on the band, or more generally a content-zero subset. Then for any continuous function \(f : \mathbb{R} \to \mathbb{R}\),
\[
\iint_{\mathcal{B}} f \, dA = \iiint_{\tilde{J}} f \circ \tilde{G} \cdot \left| J_{\tilde{G}} \right| dA \tag{*}
\]
\[
\iint_{\mathcal{B}} f(x, y) \, dx \, dy = \iiint_{\tilde{J}} f(\tilde{G}(u, v)) \, det(J_{\tilde{G}}(u, v)) \, du \, dv
\]
where
\[
J_{\tilde{G}} = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial (x, y)}{\partial (u, v)}
\end{pmatrix}
\]
is the Jacobian or total derivative of \(\tilde{G}\).

We now give an informal explanation of how \(x(y)\) is obtained.
Suppose \( \tilde{G} : \tilde{T} \rightarrow \tilde{B} \) is a 1-to-1 /onto, \( C^1 \) map, sending 
\((u,v) \mapsto \tilde{G}(u,v) = (x(u,v), y(u,v)) \).

Then for any continuous function
\( f : \tilde{B} \rightarrow \mathbb{R} \),

\[
\iint_{\tilde{B}} f \, dA = \iint_{\tilde{T}} f \circ \tilde{G} \cdot \left| J_{\tilde{G}} \right| \, dA \tag{*}
\]

\[
\iint_{\tilde{B}} f(x,y) \, dx \, dy = \iint_{\tilde{T}} f(\tilde{G}(u,v)) \, det(J_{\tilde{G}}(u,v)) \, du \, dv
\]

where
\[
J_{\tilde{G}} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial (x,y)}{\partial (u,v)} \end{pmatrix}
\]

is the Jacobian or total derivative of \( \tilde{G} \).

We now give an informal explanation of how \((x)\) is obtained.
Suppose $\tilde{G}: T \to S$ is a 1-to-1 onto, $C^1$ map, sending $(u,v) \mapsto \tilde{G}(u,v) = (x(u,v), y(u,v))$. Then for any continuous function $f: S \to \mathbb{R}$,

$$\int_S f \, dA = \int_T f \circ \tilde{G} \cdot |J_{\tilde{G}}| \, dA \quad (\star)$$

and

$$\int_S f(x,y) \, dx \, dy = \int_T f(\tilde{G}(u,v)) \, det(J_{\tilde{G}}(u,v)) \, du \, dv$$

where

$$J_{\tilde{G}} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial x(u,v)}{\partial v} \end{pmatrix}$$

is the Jacobian or total derivative of $\tilde{G}$.

We now give an informal explanation of how $(\star)$ is obtained.
Suppose \( \tilde{G}: \tilde{T} \rightarrow \delta \) is a 1-to-1 onto, \( C^1 \) map, sending \( (u,v) \rightarrow \tilde{G}(u,v) = (x(u,v), y(u,v)). \)

Then for any continuous function \( f: \delta \rightarrow \mathbb{R} \),

\[
\iint_\delta f \, d\delta = \iint_\tilde{T} f \circ \tilde{G} \cdot |J_{\tilde{G}}| \, dA \tag{\star}
\]

\[
\iint_\delta f(x,y) \, dx \, dy = \iint_\tilde{T} f(\tilde{G}(u,v)) \det(J_{\tilde{G}}(u,v)) \, du \, dv
\]

where

\[
J_{\tilde{G}} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial (x,y)}{\partial (u,v)} \end{pmatrix}.
\]

is the Jacobian or total derivative of \( \tilde{G} \).

We now give an informal explanation of how \( (\star) \) is obtained.
Suppose $\vec{G} : \Omega \to \mathbb{R}$ is a 1-to-1 /onto, $C^1$ map, sending $(u,v) \mapsto \vec{G}(u,v) = (x(u,v), y(u,v))$.

Then for any continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\iiint_{\Omega} f \, dA = \iint_{\vec{G}(\Omega)} f \circ \vec{G} \cdot \det |J_{\vec{G}}| \, dA \quad (\ast)$$

$$\iint_{\Omega} f(x,y) \, dx \, dy = \iint_{\vec{G}(\Omega)} f(\vec{G}(u,v)) \det |J_{\vec{G}}(u,v)| \, du \, dv$$

where

$$J_{\vec{G}} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial (x,y)}{\partial (u,v)} \end{pmatrix}$$

is the Jacobian or total derivative of $\vec{G}$.

We now give an informal explanation of how $(\ast)$ is obtained.
Enclose $T$ in a rectangle $Q$ and partition it; for simplicity we assume $\tilde{G}$ extends to $Q$. The $G(Q_{ij})$ aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross product:

$$
P_u = \tilde{G}(u_0 + \Delta u, \nu_0) - \tilde{G}(u_0, \nu_0) = J_G(u_0, \nu_0)(\Delta u) = \begin{pmatrix}
\frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\
\frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v}
\end{pmatrix} \Delta u
$$

$$
P_v = \tilde{G}(u_0, \nu_0 + \Delta v) - \tilde{G}(u_0, \nu_0) = J_G(u_0, \nu_0)(\Delta v) = \begin{pmatrix}
\frac{\partial x(u,v)}{\partial u} & \frac{\partial x(u,v)}{\partial v} \\
\frac{\partial y(u,v)}{\partial u} & \frac{\partial y(u,v)}{\partial v}
\end{pmatrix} \Delta v
$$

$$
a(\tilde{G}(Q_{ij})) = \left\| P_u \times P_v \right\| = \left\| 
\begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix}
\right\|
$$

Suppose $\tilde{G}: T \to \mathcal{R}$ is a 1-to-1 /onto, $C^1$ map, sending $(u,v) \mapsto \tilde{G}(u,v) = (x(u,v), y(u,v))$. Then for any continuous function $f: \mathcal{R} \to \mathbb{R}$,

$$
\int_T f \, dA = \int_T f \circ \tilde{G} \cdot \left| J_{\tilde{G}} \right| \, dA \quad \text{(\*)}
$$

\[
\int_T f(x,y) \, dx \, dy = \int_T f(\tilde{G}(u,v)) \det(J_{\tilde{G}}(u,v)) \, du \, dv
\]

where

$$
J_{\tilde{G}} = \begin{pmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{pmatrix}
$$

is the Jacobian or total derivative of $\tilde{G}$. We now give an informal explanation of how $(x')$ is obtained.

\text{Some writing also shown.}
Enclose \( T \) in a rectangle \( Q \) and partition it.

For simplicity we assume \( \hat{G} \) extends to \( G \).

The \( \hat{G}(Q_{ij}) \) aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross product:

\[
\mathbf{P}_u = \hat{G}(u_0, v_0 + \Delta u, v_0) - \hat{G}(u_0, v_0) = \frac{\partial G(u_0, v_0)}{\partial u} \Delta u
\]

\[
\mathbf{P}_v = \hat{G}(u_0, v_0, v_0 + \Delta v) - \hat{G}(u_0, v_0) = \frac{\partial G(u_0, v_0)}{\partial v} \Delta v
\]

\[
a(\hat{G}(Q_{ij})) \approx \mathbf{P}_u \times \mathbf{P}_v = \left| \begin{array}{ccc}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}
\end{array} \right| \Delta u \Delta v = \left| \begin{array}{ccc}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}
\end{array} \right| \Delta u \Delta v = \left| J \hat{G}(u_0, v_0) \right| a(Q_{ij})
\]
Enclose $\mathcal{I}$ in a rectangle $Q$ and partition it;

$$
(w_0, y_0 + n\Delta y, w_0 + n\Delta w, y_0 + n\Delta y)
$$

for simplicity we assume $\mathcal{I}$ extends to $Q$.

The $\mathcal{G}(Q_{ij})$ aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross product:

$$
\mathbf{p}_w := \mathcal{G}(w_0 + \Delta w, y_0) - \mathcal{G}(w_0, y_0) = \dot{J}_{w}(w_0, y_0)(\Delta w) = \begin{vmatrix}
\Delta w \\
\frac{\partial x}{\partial w}
\end{vmatrix}
$$

$$
\mathbf{p}_v := \mathcal{G}(w_0, y_0 + \Delta v) - \mathcal{G}(w_0, y_0) = \dot{J}_{v}(w_0, y_0)(\Delta v) = \begin{vmatrix}
\Delta v \\
\frac{\partial x}{\partial v}
\end{vmatrix}
$$

$$
a(\mathcal{G}(Q_{ij})) \approx \left\| \mathbf{p}_w \times \mathbf{p}_v \right\| = \left\| \begin{vmatrix}
1 & 2 & 3 \\
\Delta x & \Delta y & \Delta z \\
\Delta w & \Delta v & \Delta w
\end{vmatrix} \right\|
$$

= \begin{vmatrix}
\frac{\partial (x, y, z)}{\partial (w, v)} & \frac{\partial (x, y, z)}{\partial (w, v)} & \frac{\partial (x, y, z)}{\partial (w, v)} \\
\Delta w & \Delta v & \Delta w
\end{vmatrix} = |\dot{J}_{w}(w_0, y_0)| a(Q_{ij})
$$

So $|\dot{J}_{w}|$ evaluated at a point of $Q_{ij}$ approximates the dilation factor $a(\mathcal{G}(Q_{ij})) / a(Q_{ij})$. 
Enclose \( \tilde{\Omega} \) in a rectangle \( Q \) and partition it;

\[
(u_0, v_0) + n \Delta u \times (u_0, v_0) = \tilde{G}(u_0, v_0)(\Delta u) = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}
\]

for simplicity we assume \( \tilde{\Omega} \) extends to \( Q \).

The \( \tilde{G}(Q_{ij}) \) aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross product:

\[
P_u := \tilde{G}(u_0 + \Delta u, v_0) - \tilde{G}(u_0, v_0) = J_{\tilde{G}}(u_0, v_0)(\Delta u)
\]

\[
P_v := \tilde{G}(u_0, v_0 + \Delta v) - \tilde{G}(u_0, v_0) = J_{\tilde{G}}(u_0, v_0)(\Delta v)
\]

\[
a(\tilde{G}(Q_{ij})) \approx \|P_u \times P_v\| = \begin{vmatrix}
E_1 & E_2 & E_3 \\
\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\
\frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0
\end{vmatrix}
\]

\[= \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix} \Delta u \Delta v = |J_{\tilde{G}}(u_0, v_0)| a(Q_{ij}).
\]

So \( |J_{\tilde{G}}| \) evaluated at a point of \( Q_{ij} \) approximates the dilation factor \( \frac{a(\tilde{G}(Q_{ij}))}{a(Q_{ij})} \), and

\[
\int_I^\delta f \, dA = \lim_{\|Q_{ij}\| \to 0} \sum_k f(\tilde{G}(Q_{ij})) a(\tilde{G}(Q_{ij}))
\]

(as the "norm of the partition", i.e. minimum side-length of a \( Q_{ij} \), goes to 0)

\[
= \lim_{\|Q_{ij}\| \to 0} \sum_k \int_{Q_{ij}} f(\tilde{G}(Q_{ij})) |J_{\tilde{G}}(Q_{ij})| a(Q_{ij})
\]

\[
= \int_\Omega \left( f \circ \tilde{G} \right) |J_{\tilde{G}}| \, dA.
\]
Enclose $J$ in a rectangle $Q$ and partition it;

$$\| \vec{r}_u \times \vec{r}_v \| = \left| \begin{array}{ccc} 1 & 2 & 3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{array} \right|$$

for simplicity we assume $G$ extends to $Q$.

The $G(Q_{ij})$ aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross product:

$$P_u := \vec{r}_u \times \vec{r}_v = G(u, v) \hat{u} \times \hat{v} = J_0(u, v) \left( \frac{\partial u}{\partial u} \hat{u} + \frac{\partial v}{\partial v} \hat{v} \right) = \frac{\partial u}{\partial u} \hat{u} \times \frac{\partial v}{\partial v} \hat{v}$$

So $|J_0|$ evaluated at a point of $Q_{ij}$ approximates the dilation factor $\frac{a(G(Q_{ij}))}{a(Q_{ij})}$, and

$$\iint_{Q_{ij}} f \, dA = \lim_{\Delta u \to 0} \sum_{ij} f(G(Q_{ij})) \Delta (Q_{ij})$$

as the “norm of the partition” (i.e., maximum side-length of a $Q_{ij}$) goes to 0.

$$= \lim_{\Delta u \to 0} \sum_{ij} f(G(Q_{ij})) |J_0(Q_{ij})| a(Q_{ij})$$

$$= \iint_{Q} f(G(Q)) \, J_0 \, dA.$$ 

A key special case which we shall now focus on is when $(u, v)$ is called $(r, \theta)$ and $G(r, \theta) = (r \cos \theta, r \sin \theta).$
So \(|J_6|\) evaluated at a point of \(Q_{ij}\) approximates the dilation factor \(\frac{a(G(Q_{ij}))}{a(Q_{ij})}\), and 

\[
\int f \, dA \approx \lim_{\text{partition size goes to 0}} \sum_{ij} f(G(Q_{ij}))a(G(Q_{ij}))
\]

as the “norm of the partition,” i.e., minimum side-length of a \(Q_{ij}\) goes to 0.

\[
= \lim_{\text{partition size goes to 0}} \sum_{ij} (f \circ G)(Q_{ij}) |J_6(Q_{ij})|a(Q_{ij})
\]

\[
= \int f \circ G \cdot |J_6| \, dA.
\]

A key special case which we shall now focus on is when \((u,v)\) is called \((r,\theta)\) and \(G(r,\theta) = (r \cos \theta, r \sin \theta)\). The Jacobian determinant is

\[
|J_6| = \begin{vmatrix}
\frac{\partial}{\partial r}(r \cos \theta) & \frac{\partial}{\partial \theta}(r \cos \theta) \\
\frac{\partial}{\partial r}(r \sin \theta) & \frac{\partial}{\partial \theta}(r \sin \theta)
\end{vmatrix}
= \begin{vmatrix}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{vmatrix}
\]

\[
= r \cos \theta (\cos \theta) - (-r \sin \theta) \sin \theta
= r(\cos^2 \theta + \sin^2 \theta) = r.
\]
and so the change-of-variable formula is

\[ \int_S f(x,y) \, dx \, dy = \int_{S'} (f \circ \phi) (\rho, \theta) \, \rho \, d\rho \, d\theta. \]

So \(|J_6|\) evaluated at a point of \(Q_{ij}\) approximates the dilation factor \(\frac{a(\vec{G}(Q_{ij}))}{a(Q_{ij})}\), and

\[ f(dA) = \lim_{ij \to 0} \frac{1}{\rho_{ij}} \int_{\rho_{ij}} (f \circ \phi') (\rho, \theta) \, J_6 |J_6| \, dA. \]

A key special case which we shall now focus on is when \((u,v)\) is called \((r,\theta)\) and \(\vec{G}(r,\theta) = (r \cos \theta, r \sin \theta)\).

The Jacobian determinant is

\[ |J_6| = \begin{vmatrix} \frac{\partial x}{\partial r} (r \cos \theta) & \frac{\partial x}{\partial \theta} (r \sin \theta) \\ \frac{\partial y}{\partial r} (r \cos \theta) & \frac{\partial y}{\partial \theta} (r \sin \theta) \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta - (-r \sin \theta) \sin \theta = r (\cos^2 \theta + \sin^2 \theta) = r, \]
and so the change of variable formula is
\[ \iint f(x,y) \, dx \, dy = \iint (f \circ \tilde{G})(r,\theta) \, r \, dr \, d\theta. \]

Intuitively, this is replacing the Riemann sum over rectangles by one over angular sectors
\[ \text{area} = r \, dr \, d\theta \]
which, after all, are the \( \tilde{G}(Q_{ij}) \) in this case.

So \( |J_6| \) evaluated at a point of \( Q_{ij} \) approximates the dilation factor \( \frac{a(G(Q_{ij}))}{a(Q_{ij})} \), and
\[ \iint r \, dr \, d\theta = \lim_{\text{point in } Q_{ij} \to \text{ point in } Q_{ij}} \sum_{ij} f(\tilde{G}(Q_{ij})) \cdot a(G(Q_{ij})) \]

\[ \cdot \text{“norm of the partition,”}
\]
\[ \text{i.e. minimum side length of a } Q_{ij}, \text{ goes to } 0 \]
\[ = \lim_{\text{point in } Q_{ij} \to \text{ point in } Q_{ij}} \sum_{ij} (f \circ \tilde{G})'(Q_{ij}) \cdot |J_6(Q_{ij})| \cdot a(Q_{ij}) \]
\[ = \int_{Q_{ij}} (f \circ \tilde{G})' \, dA. \]

A key special case which we shall now focus on is where \((u,v)\) is called \((r,\theta)\) and 
\[ \tilde{G}(r,\theta) = (r \cos \theta, r \sin \theta). \]
The Jacobian determinant is
\[ |J_6| = \begin{vmatrix} \frac{\partial (r \cos \theta)}{\partial r} & \frac{\partial (r \cos \theta)}{\partial \theta} \\ \frac{\partial (r \sin \theta)}{\partial r} & \frac{\partial (r \sin \theta)}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \]
\[ = r \cos^2 \theta - (-r \sin \theta) \sin \theta \]
\[ = r (\cos^2 \theta + \sin^2 \theta) = r, \]
and so the change-of-variable formula is
\[
\int \int f(x,y) \, dx \, dy = \int \int (f(r\cos \theta, r\sin \theta)) \, r \, dr \, d\theta.
\]

Intuitively, this is replacing the Riemann sum over rectangles by one over angular sectors,
\[\text{area} = r \, dr \, d\theta,\]
which, after all, are the \(O(0;ij)\) in this case.

We now turn to several examples of polar integration.

Ex 3: Find the volume of the solid in the 1st octant bounded by \(z = x^2 + y^2, \ x^2 + y^2 = 4,\) and the coordinate planes.
and so the change-of-variable formula is
\[ \int \int f(x,y) \, dx \, dy = \int \int (f \circ \theta)(r, \theta) \, r \, dr \, d\theta. \]

Intuitively, this is replacing the Riemann sum over rectangles by one over angular sectors:
\[ \text{area} = r \, dr \, d\theta \]

which, after all, are the \( \theta \)-circles in this case.

We now turn to several examples of
gold in the integration.

Ex 3 | Find the volume of
the solid in the 1st octant
bounded by \( z = x^2 + y^2 \),
\( x^2 + y^2 = 4 \), and the coordinate planes.
\[ V = \iiint_D (x^2 + y^2) \, dA \]
and so the change-of-variable formula is

$$\int_{\Omega} f(x,y) \, dx \, dy = \int_{\Sigma} f(\phi^{-1}(r,\theta)) \, r \, dr \, d\theta.$$  

Intuitively, this is replacing the Riemann sum over rectangles by one over angular sectors

$$\text{area} = r \, dr \, d\theta$$

which, after all, are the $\{Q_{ij}\}$ in this case.

We now turn to several examples of polar integration.

**Ex 3** Find the volume of the solid in the 1st octant bounded by $z = x^2 + y^2$, $x^2 + y^2 = 4$, and the coordinate planes.

$$V = \iint_{\Sigma} (x^2 + y^2) \, dA$$

$$= \int_0^{2\pi} \int_0^2 (r^2 \cos^2 \theta + r^2 \sin^2 \theta) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left( \frac{1}{2} r^2 \cos^2 \theta + \frac{1}{2} r^2 \sin^2 \theta \right) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \left[ \frac{8}{3} \cos^3 \theta + 8 \sin^3 \theta \cos \theta \right] \, d\theta$$

$$= 2\pi \left[ \frac{16}{3} \sin^4 \theta + 16 \sin^5 \theta \cos \theta \right]_0^{2\pi}$$

$$= 2\pi.$$

... blood, sweat, tears
and so the change-of-variable formula is
\[
\int \int_D f(x,y) \, dx \, dy = \int \int_{D'} f(\rho \cos \theta, \rho \sin \theta) \rho \, dr \, d\theta.
\]

Intuitively, this is replacing the Riemann sum over rectangles by one over angular sectors:
\[
\text{area} = rdrd\theta
\]

which, after all, are the \( B(Q_{ij}) \) in this case.

We now turn to several examples of polar integration.

Ex 3 Find the volume of the solid in the 1st octant bounded by \( z = x^2 + y^2 \), \( x^2 + y^2 = 4 \), and the coordinate planes.

\[
V = \int \int_D (x^2 + y^2) \, dA
\]
\[
= \int \int_D \left( (\rho \cos \theta)^2 + (\rho \sin \theta)^2 \right) \rho \, dr \, d\theta
\]

\[\leftarrow \text{what is this?}\]
and so the change-of-variables formula is
\[ \iint f(x,y) \, dx \, dy = \int_0^2 \int_0^\pi (r^2 \theta)(r,\theta) \, r \, dr \, d\theta. \]

Intuitively, this is replacing the Riemann sum over rectangles by one over angular sectors
\[ \text{area} = r \, dr \, d\theta \]
which, after all, are the \([\theta \, (\theta,\theta)]\) in this case.

We now turn to several examples of polar integration.

Ex 3 | Find the volume of the solid in the 1st octant bounded by \( z = x^2 + y^2 \), \( x^2 + y^2 = 4 \), and the coordinate planes.

\[ V = \iiint_A (x^2 + y^2) \, dA \]
\[ = \iiint_A \left( r \cos^2 \theta + r^2 \sin^2 \theta \right) \, r \, dr \, d\theta \]
\[ = \int_0^{\pi/2} \left( \int_0^2 r^3 \, dr \right) \, d\theta \]
\[ = \int_0^{\pi/2} \frac{2^4}{4} \, d\theta = 4 \cdot \frac{\pi}{2} = 2\pi. \]

No sweat.
Ex 3 Find the volume of the solid in the 1st octant bounded by \( z = x^2 + y^2 \), \( x^2 + y^2 = 4 \), and the coordinate planes.

\[
V = \iiint_{\mathcal{D}} (x^2 + y^2)\,dA
\]

\[
= \iint_{\mathcal{R}} \left( r^2 \cos^2 \theta + r^2 \sin^2 \theta \right) \,r\,dr\,d\theta
\]

\[
= \int_{0}^{2\pi} \left( \int_{0}^{2} r^2 \,dr \right) \,d\theta
\]

\[
= \int_{0}^{2\pi} \frac{2^4}{4} \,d\theta = \frac{16}{4} \cdot \frac{\pi}{2} = 2\pi
\]

No sweat.
Ex 4 \[ \int \int_D f(x,y) \, dx \, dy = \int \int_D (R^2)(r^2) \, r \, dr \, d\theta \]

Ex 3 Find the volume of the solid in the 1st octant bounded by \( z = x^2 + y^2 \), \( x^2 + y^2 = 4 \), and the coordinate planes.

\[ V = \iiint_D (x^2 + y^2) \, dV \]
\[ = \iiint_D \left( (r \cos \theta)^2 + (r \sin \theta)^2 \right) r \, dr \, d\theta \]
\[ = \int_0^\sqrt{2} \int_0^\pi \left( \int_0^2 r^3 \, dr \right) \, d\theta \]
\[ = \int_0^\pi \left( \frac{4}{4} \right) \, d\theta = 4 \cdot \frac{\pi}{2} = 2\pi. \]

No sweat.
**Ex 4** Compute the volume of a sphere of radius $R$. Then find the volume of a cored apple, if the apple is a sphere of radius $R$ and the core is its intersection with a solid cylinder of radius $c$.

In both cases, you have to integrate the difference between the top and the bottom of the "apple":

$$V = \iiint_{S} \left( \sqrt{R^2 - x^2 - y^2} - (-\sqrt{R^2 - x^2 - y^2}) \right) \, dA$$

$$= 2 \int_{0}^{2\pi} \int_{0}^{R} \sqrt{R^2 - r^2} \, r \, dr \, d\theta$$

**Ex 3** Find the volume of the solid in the 1st octant bounded by $z = x^2 + y^2$, $x^2 + y^2 = 4$, and the coordinate planes.

$$V = \iiint_{S} (x^2 + y^2) \, dA$$

$$= \iiint_{S} \left( (r \cos \theta)^2 + (r \sin \theta)^2 \right) \, r \, dr \, d\theta$$

$$= \int_{0}^{\pi/2} \int_{0}^{2} r^3 \, dr \, d\theta$$

$$= \int_{0}^{\pi/2} \frac{2}{4} \, d\theta = 4 \cdot \frac{\pi}{2} = 2\pi.$$
Ex 4] Compute the volume of a sphere of radius $R$.
Then find the volume of a cored apple, if the apple is a sphere of radius $R$ and the core is its intersection with a solid cylinder of radius $c$.

In both cases, you have to integrate the difference between the top and the bottom of the "apple":

$$V = \iiint_{B} \left\{ \sqrt{R^2-x^2-y^2} - \left( -\sqrt{R^2-x^2-y^2} \right) \right\} \, dA$$
$$= 4\pi \int_{0}^{R} \sqrt{R^2-r^2} \, r \, dr$$

Ex 3] Find the volume of the solid in the 1st quadrant bounded by $z = x^2+y^2$, $x^2+y^2 = 4$, and the coordinate planes.

$$V = \iiint_{S} (x^2+y^2) \, dA$$
$$= \iiint_{S} \left( (r \cos \theta)^2 + (r \sin \theta)^2 \right) \, r \, dr \, d\theta$$
$$= \int_{0}^{\pi/2} \int_{0}^{2} \frac{r^4}{4} \, dr \, d\theta$$
$$= \int_{0}^{\pi/2} \frac{2^5}{2} \, d\theta = 4 \cdot \frac{\pi}{2} = 2\pi.$$
Ex 4. Compute the volume of a sphere of radius $R$.
Then find the volume of a cored apple, if the apple is a sphere of radius $R$ and the core is its intersection with a solid cylinder of radius $c$.

In both cases, you have to integrate the difference between the top and the bottom of the "apple":

$$ V = \iiint_B \left( \sqrt{R^2-x^2-y^2} - \left(-\sqrt{R^2-x^2-y^2}\right) \right) \, dA $$

$$ = 4\pi \int_c^R \sqrt{R^2-r^2} \, r \, dr $$

$$ = 4\pi \left[ -\frac{1}{3} (R^2-r^2)^{3/2} \right]_c^R = \frac{4}{3} \pi (R^2-c^2)^{3/2}. $$

Ex 3. Find the volume of the solid in the 1st octant bounded by $z = x^2+y^2$, $x^2+y^2=4$, and the coordinate planes.

$$ V = \iiint_S (x^2+y^2) \, dA $$

$$ = \int_0^\pi \left( \int_1^2 (r^2 \cos^2 \theta + r^2 \sin^2 \theta) \right) \, r \, dr \, d\theta $$

$$ = \int_0^\pi \left( \int_1^2 r^3 \, dr \right) \, d\theta $$

$$ = \int_0^\pi \frac{2}{4} \, d\theta = 4 \cdot \frac{\pi}{2} = 2\pi. $$

No sweat.
Ex 4] Compute the volume of a sphere of radius \( R \).

Then find the volume of a cored apple, if the apple is a sphere of radius \( R \) and the core is its intersection with a solid cylinder of radius \( c \).

In both cases, you have to integrate the difference between the top and the bottom of the "apple":

\[
V = \iint_{\mathcal{B}} \left( \sqrt{R^2-x^2-y^2} - \left( -\sqrt{R^2-x^2-y^2} \right) \right) \, dA
\]

\[
= 4\pi \int_{c}^{R} \sqrt{R^2-r^2} \, r \, dr
\]

\[
= 4\pi \left[ -\frac{1}{3} (R^2-r^2)^{3/2} \right]_{c}^{R} = \frac{4}{3} \pi (R^2-c^2)^{3/2}
\]

\[
= \frac{4}{3} \pi h^3, \quad h = \text{height of cored apple} \quad (h = R \text{ if not cored})
\]
Ex 4] Compute the volume of a sphere of radius $R$.
Then find the volume of a cored apple, if the apple is a sphere of radius $R$ and the core is its intersection with a solid cylinder of radius $c$.

In both cases, you have to integrate the difference between the top and the bottom of the "apple":

\[
V = \iiint_S \left[ \sqrt{R^2 - x^2 - y^2} - \left( -\sqrt{R^2 - x^2 - y^2} \right) \right] \, dA
= 4\pi \int_c^R \sqrt{R^2 - r^2} \, r \, dr
= 4\pi \left[ -\frac{1}{3} (R^2 - r^2) \right]_c^R = \frac{4}{3}\pi (R^2 - c^2)^{3/2}
= \frac{4}{3}\pi h^3, \quad h = \text{height of cored apple} (h = R \text{ if not cored})
\]

Ex 5] Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx$.
This is nasty to work with directly in $x$ or $y$.
Convert to polar coordinates:

\[
y = \sqrt{2x-x^2} \quad \Rightarrow \quad y^2 = 2x-x^2
\]
\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
(x-1)^2 + y^2 &= 1
\end{align*}
\]
Ex 4. Compute the volume of a sphere of radius $R$.
Then find the volume of a cored apple, if the apple is a sphere of radius $R$ and the core is its intersection with a solid cylinder of radius $C$.

In both cases, you have to integrate the difference between the top and the bottom of the “apple”:

$$V = \int_S \left( \sqrt{R^2-x^2-y^2} - (-\sqrt{R^2-x^2-y^2}) \right) dA$$

$$= 4\pi \int_c^R \sqrt{R^2-r^2} \ r \ dr$$

$$= 4\pi \left[ \frac{1}{3} (R^2-r^2)^{3/2} \right]_c^R = \frac{4}{3}\pi (R^2-C^2)^{3/2}$$

$$= \frac{4}{3}\pi \ h^3 \quad , \quad h= \text{height of cored apple}$$

Ex 5. Evaluate \( \int_0^2 \int_0^{\sqrt{2-x}} \sqrt{x^2+y^2} \ dy \ dx \).
This is nasty to use differentiation in $x$ or $y$.
Convert to polar coordinates:

$$\int_0^\pi \int_0^R r^2 \cos^2 \theta \ dr \ d\theta$$
Ex 6. Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

(essentially the bell curve)

\[
\int_{-\infty}^{\infty} f(x,y) \, dx \, dy = \int_0^\infty \int_0^{2\pi} \rho^2 f(\rho\cos\theta, \rho\sin\theta) \, d\rho \, d\theta
\]

Ex 5. Evaluate \( \int_0^\infty \int_0^\infty \sqrt{2x-x^2} \, dy \, dx \).

This is nasty to use differentials in \( x \) or \( y \).

Convert to polar coordinates:

\[
y = \sqrt{2x-x^2} \\
y^2 = 2x-x^2 \\
(x-1)^2 + y^2 = 1 \\
x^2 + y^2 = 2x \\
r^2 = 2r \cos \theta \\
r = 2 \cos \theta \\
\]

\[
\int_0^\infty \int_0^\infty \rho^2 f(\rho\cos\theta, \rho\sin\theta) \, d\rho \, d\theta
\]

\[
= \frac{8}{3} \int_0^{\pi/2} \cos^3 \theta \, d\theta = \frac{8}{3} \int_0^{\pi/2} (\cos \theta - \sin^2 \theta \cos \theta) \, d\theta
\]

\[
= \frac{8}{3} \left[ \sin \theta - \frac{1}{3} \sin^3 \theta \right]_0^{\pi/2} = \frac{8}{3} \left( 1 - \frac{1}{3} \right) = \frac{16}{9}.
\]
Ex 6: Compute $I = \int_{-\infty}^{\infty} e^{-x^2} \, dx$.

Consider the volume $V$ under $e^{-x^2-y^2}$ in two ways:

Ex 5: Evaluate $\int_{0}^{2} \int_{0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx$.

This is messy to use differentiation in $x$ or $y$.

Convert to polar coordinates:

\[
\int_{0}^{2\pi} \int_{0}^{r} \sqrt{r^2} \, r \, dr \, d\theta = \int_{0}^{\frac{\pi}{2}} \int_{0}^{r} \sqrt{r^2} \, r \, dr \, d\theta
\]

\[
= \frac{8}{3} \int_{0}^{\frac{\pi}{2}} \cos^3 \theta \, d\theta = \frac{8}{3} \int_{0}^{\frac{\pi}{2}} (\cos \theta - \sin^2 \theta \cos \theta) \, d\theta
\]

\[
= \frac{8}{3} \left[ \sin \theta - \frac{1}{3} \sin^3 \theta \right]_{0}^{\frac{\pi}{2}} = \frac{8}{3} (1 - \frac{1}{3}) = \frac{16}{9}.
\]
Ex 6. Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

Consider the volume \( V \) under \( e^{-x^2-y^2} \) in two ways:

1. \[ V = \lim_{b \to \infty} \int_{-b}^{b} \int_{-b}^{b} e^{-x^2-y^2} \, dy \, dx \]

\[ = \lim_{b \to \infty} \left( \int_{-b}^{b} e^{-x^2} \, dx \right) \left( \int_{-b}^{b} e^{-y^2} \, dy \right) \]

\[ = \left( \lim_{b \to \infty} \int_{-b}^{b} e^{-x^2} \, dx \right)^2 = I^2. \]

\[ \iint_{S} f(x,y) \, dx \, dy = \iiint_{S} (f \circ \mathbf{g})(r,\theta) \, r \, dr \, d\theta \]
Ex 6: Compute 
\[ I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \.
\]

Consider the volume \( V \) under \( e^{-x^2 - y^2} \) in two ways:

1. \[ V = \lim_{b \to \infty} \int_{-b}^{b} \int_{-b}^{b} e^{-x^2 - y^2} \, dy \, dx \]
   \[ = \lim_{b \to \infty} \left( \int_{-b}^{b} e^{-x^2} \, dx \right) \left( \int_{-b}^{b} e^{-y^2} \, dy \right) \]
   \[ = \left( \lim_{b \to \infty} \int_{-b}^{b} e^{-x^2} \, dx \right)^2 = \pi^2. \]

2. \[ V = \int_0^{2\pi} \int_0^\infty f(x, y) \, dx \, dy = \int_0^{2\pi} \int_0^\infty \theta \, r \, dr \, d\theta \]
   \[ = \int_0^{2\pi} \left[ \frac{1}{2} e^{-r^2} \right]^a_0 \, d\theta \]
   \[ = \lim_{a \to \infty} \frac{1}{2} \int_0^{2\pi} \left( 1 - e^{-a^2} \right) \, d\theta \]
   \[ = \pi \left( 1 - e^{-a^2} \right). \]
Ex 6: Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

Consider the volume \( V \) under \( e^{-x^2-y^2} \) in two ways:

1. \( V = \lim_{b \to \infty} \int_{-b}^{b} \int_{-b}^{b} e^{-x^2-y^2} \, dy \, dx 
   = \lim_{b \to \infty} \left( \int_{-b}^{b} e^{-x^2} \, dx \right) \left( \int_{-b}^{b} e^{-y^2} \, dy \right) 
   = \left( \lim_{b \to \infty} \int_{-b}^{b} e^{-x^2} \, dx \right)^2
   = I^2. \)

2. \( V = \int_{0}^{\infty} \int_{0}^{2\pi} \frac{1}{2} r^2 e^{-r^2/4} \, r \, d\theta \, dr 
   = \lim_{a \to \infty} \int_{0}^{\infty} \int_{0}^{2\pi} \frac{1}{4} r^2 e^{-r^2/4} \, r \, d\theta \, dr 
   = \lim_{a \to \infty} \frac{1}{4} \int_{0}^{\infty} \left( 1 - e^{-a^2} \right) \, dr 
   = \lim_{a \to \infty} \pi \left( 1 - e^{-a^2} \right) 
   = \pi. \)

So, \( I^2 = \pi \Rightarrow I = \sqrt{\pi}. \)