Lecture 40: Change of Variable

Though I tend not to write it out explicitly, we have been doing a lot of substitutions in the integrals we compute. Recall how this goes: given

$$u \longmapsto g(u) = x \qquad C^1 \text{ function}$$

mapping

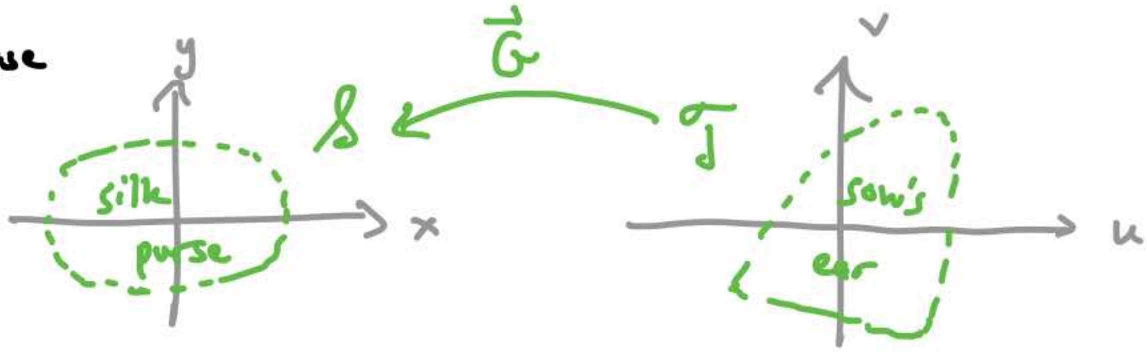
$$[c,d] \longrightarrow [a,b] \qquad a = g(c),\ b = g(d)$$

in 1-to-1 fashion,

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du.$$

We are after a 2-variable analogue of this.

Suppose



is a 1-to-1 & onto C^1 map. We'll write

$$(u,v) \longmapsto \vec{G}(u,v) = (x(u,v),\ y(u,v)),$$

and not worry about $\vec{G}$ failing to be 1-to-1 on the boundary or more generally on a content-zero subset. Then for any continuous function $f: \mathcal{S} \to \mathbb{R}$,

$$\boxed{\iint_{\mathcal{S}} f \, dA = \iint_{\mathcal{T}} (f \circ \vec{G}) \left| \det(J_{\vec{G}}) \right| dA} \quad (*)$$

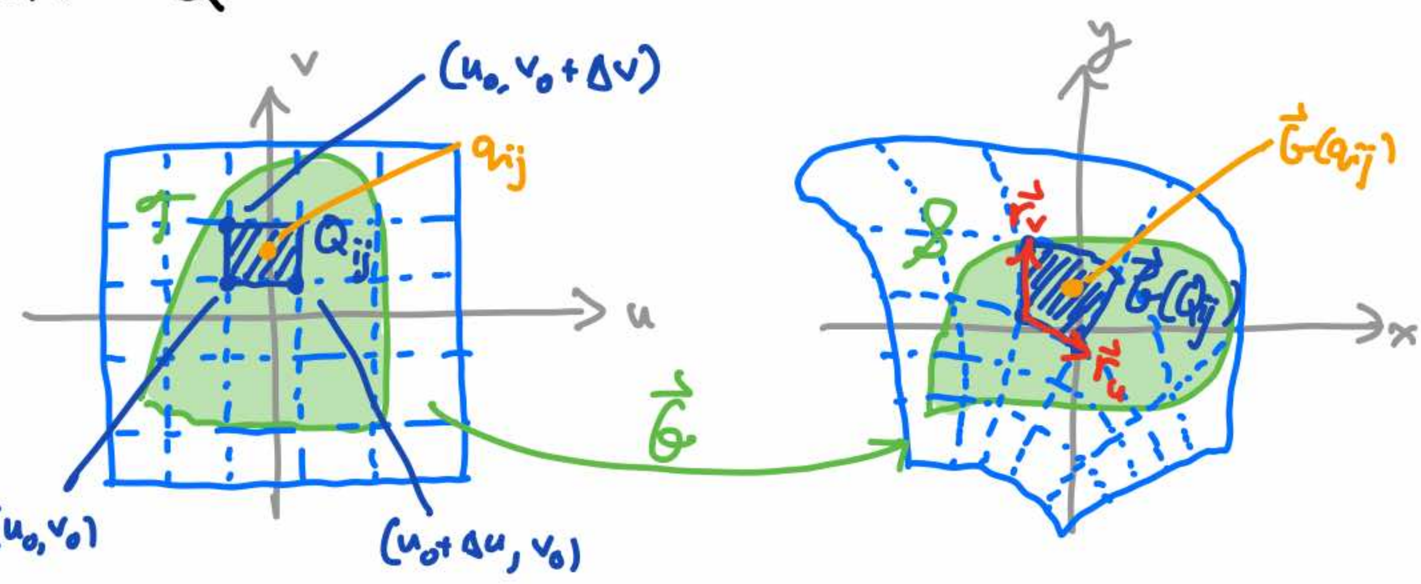
$$\iint_{\mathcal{S}} f(x,y)\,dx\,dy = \iint_{\mathcal{T}} f(\vec{G}(u,v)) \left| \det(J_{\vec{G}}(u,v)) \right| du\,dv$$

where

$$J_{\vec{G}} = \begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix} \quad \left[\overset{\text{sometimes also written}}{=} \frac{\partial(x,y)}{\partial(u,v)} \right]$$

is the Jacobian or total derivative of $\vec{G}$. We now give an informal explanation of how $(*)$ is obtained.

Begin by enclosing $\mathcal{T}$ in a rectangle Q, and partition Q:



For simplicity, we assume $\vec{G}$ extends to Q.

The $\vec{G}(Q_{ij})$ aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross product:

$$\vec{r}_u := \vec{G}(u_0+\Delta u, v_0) - \vec{G}(u_0, v_0) \approx J_{\vec{G}}(u_0,v_0)\begin{pmatrix}\Delta u\\ 0\end{pmatrix} = \begin{pmatrix}\frac{\partial x}{\partial u}(u_0,v_0)\\ \frac{\partial y}{\partial u}(u_0,v_0)\end{pmatrix}\Delta u$$

$$\vec{r}_v := \vec{G}(u_0, v_0+\Delta v) - \vec{G}(u_0, v_0) \approx J_{\vec{G}}(u_0,v_0)\begin{pmatrix}0\\ \Delta v\end{pmatrix} = \begin{pmatrix}\frac{\partial x}{\partial v}(u_0,v_0)\\ \frac{\partial y}{\partial v}(u_0,v_0)\end{pmatrix}\Delta v$$

$$a(\vec{G}(Q_{ij})) \approx \|\vec{r}_u \times \vec{r}_v\| = \left\| \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{vmatrix} \right\|$$

$$= \left|\det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{pmatrix}\right| \Delta u\, \Delta v = |\det(J_{\vec{G}}(u_0,v_0))|\, a(Q_{ij}).$$

So $|J_{\vec{G}}|$ evaluated at a point of Q_{ij} approximates the dilation factor $\dfrac{a(\vec{G}(Q_{ij}))}{a(Q_{ij})}$, and

not rigorously justified

sample point in Q_{ij}

$$\iint_{S} f\, dA = \lim_{\underbrace{\|P\|\to 0}} \sum_{i,j} f(\vec{G}(q_{ij}))\, a(\vec{G}(Q_{ij}))$$

as the "norm of the partition", i.e. maximum side-length of a Q_{ij}, goes to 0.

$$= \lim_{\|P\| \to 0} \sum_{i,j} (f \circ \vec{G})(q_{ij}) \left| \det\left(J_{\vec{G}}(q_{ij})\right) \right| a(Q_{ij})$$

$$= \iint_{\mathcal{J}} (f \circ \vec{G}) \left| \det(J_{\vec{G}}) \right| dA .$$

A key special case on which we shall now focus is where (u,v) is called (r,θ) and

$$\vec{G}(r,\theta) = (r\cos\theta,\ r\sin\theta) .$$

The Jacobian determinant is

$$\left| \det(J_{\vec{G}}) \right| = \left| \det \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\theta) & \frac{\partial}{\partial \theta}(r\cos\theta) \\ \frac{\partial}{\partial r}(r\sin\theta) & \frac{\partial}{\partial \theta}(r\sin\theta) \end{pmatrix} \right| = \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right|$$

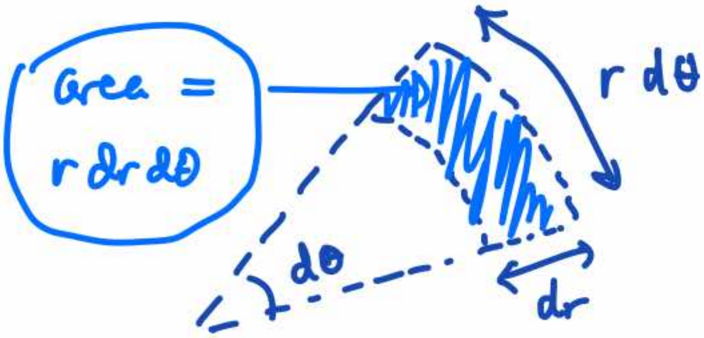
$$= r\cos\theta\,(\cos\theta) - (-r\sin\theta)\sin\theta$$

$$= r\left(\cos^2\theta + \sin^2\theta\right) = r ,$$

and so the change-of-variable formula is

$$\boxed{\iint_{\mathcal{R}} f(x,y)\, dx\, dy = \iint_{\mathcal{J}} (f \circ \vec{G})(r,\theta)\ \underline{\underline{r}}\, dr\, d\theta}$$

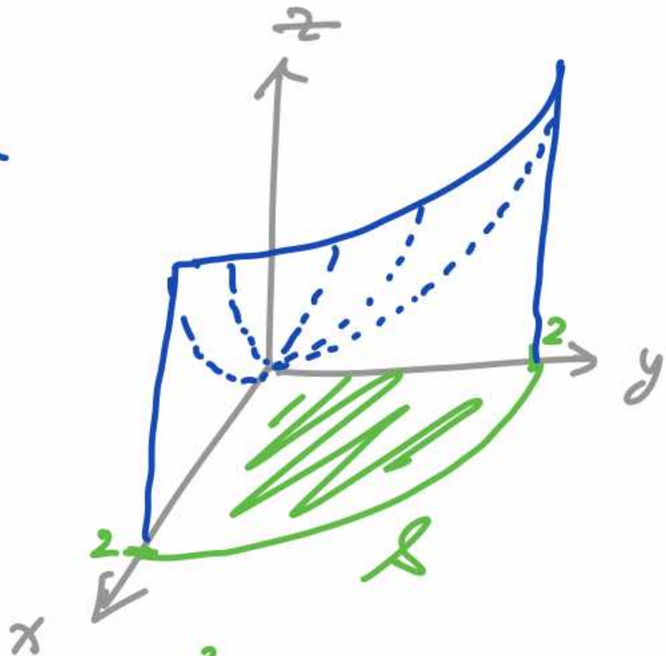
Intuitively, this is replacing the Riemann sum over rectangles by one over angular sectors —



which, after all, are the $\{\vec{G}(Q_{ij})\}$ in this case.

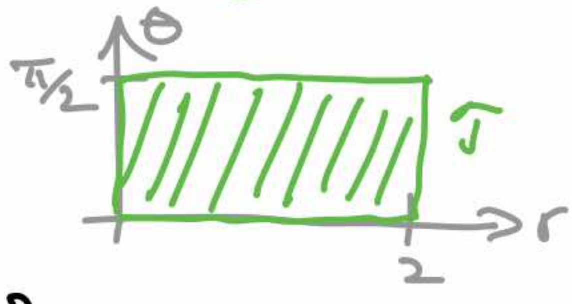
We now turn to several examples of polar integration.

Ex 1/ Find the volume of the solid in the 1^{st} octant bounded by $z = x^2+y^2$, $x^2+y^2=4$, and the coordinate planes.



$$V = \iint_{\mathcal{S}} (x^2+y^2)\,dA = \iint_{J} \Big(\overbrace{(r\cos\theta)^2+(r\sin\theta)^2}^{r^2}\Big)\, r\,dr\,d\theta$$

what is this?



$$= \int_0^{\pi/2}\left(\int_0^2 r^3\,dr\right)d\theta$$

$$= \int_0^{\pi/2} \frac{2^4}{4}\,d\theta = 4\cdot\frac{\pi}{2} = 2\pi.$$

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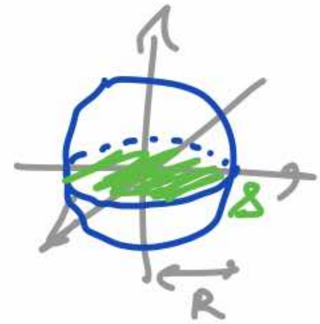
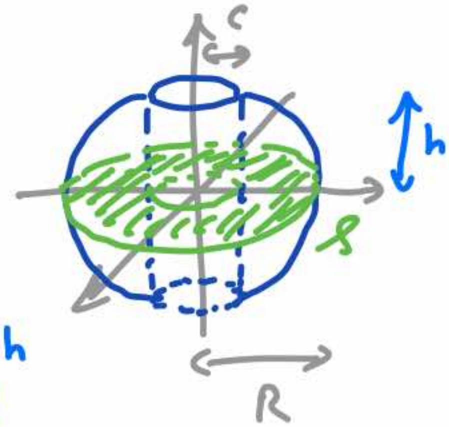
Ex 2/ Compute the volume of a sphere of radius R. Then find the volume of a cored apple, if the apple is a sphere of radius R & the core is its intersection with a solid cylinder of radius c.





In both cases, you have to integrate the difference between the top and the bottom of the "apple":

$$V = \iint_S \left\{ \sqrt{R^2-x^2-y^2} - \left(-\sqrt{R^2-x^2-y^2}\right)\right\} dA$$

$$= 2\int_0^{2\pi}\int_c^R \sqrt{R^2-r^2}\; r\,dr\,d\theta = 4\pi\int_c^R \sqrt{R^2-r^2}\; r\,dr$$

$$= 4\pi\left[-\frac{1}{3}(R^2-r^2)^{3/2}\right]_c^R = \frac{4}{3}\pi(R^2-c^2)^{3/2}$$

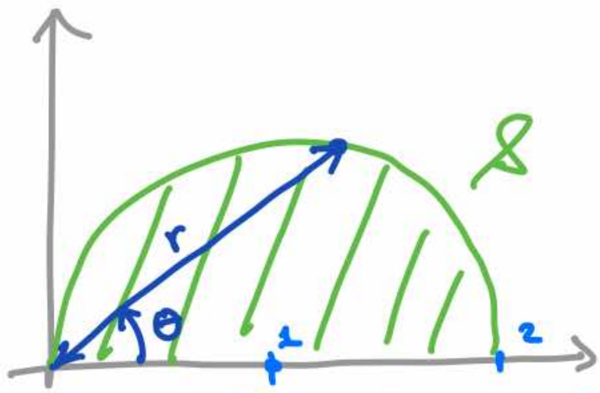
$$= \frac{4}{3}\pi h^3,$$ h = distance from center to top of cored apple

$(h = R$ if not cored$)$ //

Ex 3/ Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$.

This is nasty to antidifferentiate in x or y.

Convert it to polar coordinates:



$$\iint_{\mathcal{D}} r\, dA = \int_0^{\pi/2} \left(\int_0^{2\cos\theta} r^2\, dr \right) d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \frac{8}{3} \int_0^{\pi/2} \cos\theta \left(1 - \sin^2\theta\right) d\theta$$

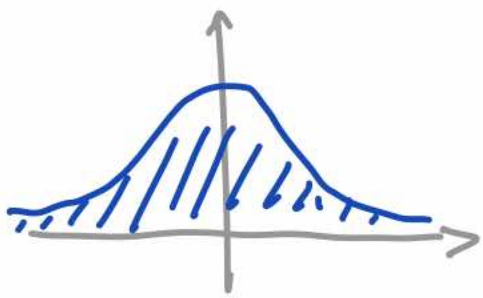
$y = \sqrt{2x-x^2} \to y^2 = 2x - x^2$

$\to (x^2-1)^2 + y^2 = 1$ or $x^2+y^2 = 2x$

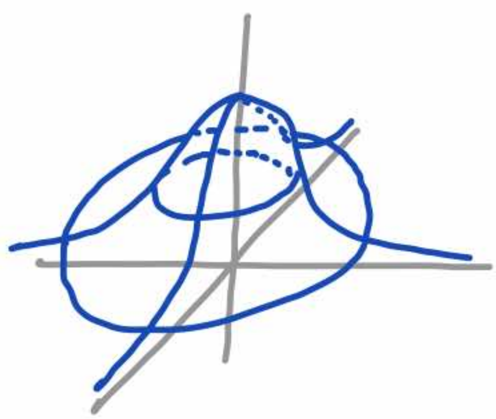
$\to r^2 = 2r\cos\theta \to \underline{r = 2\cos\theta}$.

$$= \frac{8}{3} \left[\sin\theta - \frac{\sin^3\theta}{3} \right]_0^{\pi/2} = \frac{8}{3}\left(1 - \frac{1}{3}\right) = \frac{16}{9}. \quad //$$

Ex 4/ Compute $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$.



Consider the volume V under $e^{-x^2-y^2}$:



We will compute this in two ways.

① $$V = \lim_{b\to\infty} \int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\,dy\,dx$$

$$= \lim_{b\to\infty} \left(\int_{-b}^{b} e^{-y^2}\,dy\right)\left(\int_{-b}^{b} e^{-x^2}\,dx\right)$$

$$= \left(\lim_{b\to\infty} \int_{-b}^{b} e^{-x^2}\,dx\right)^2 = I^2$$

② $$V = \lim_{a\to\infty} \int_{0}^{2\pi}\int_{0}^{a} e^{-r^2}\,r\,dr\,d\theta$$

$$= \lim_{a\to\infty} \int_{0}^{2\pi} \left[-\frac{1}{2}e^{-r^2}\right]_{r=0}^{a} d\theta$$

$$= \lim_{a\to\infty} \frac{1}{2}\int_{0}^{2\pi} \left(1-e^{-a^2}\right) d\theta$$

$$= \lim_{a\to\infty} \pi\left(1-e^{-a^2}\right) = \pi .$$

So $I^2 = \pi \Rightarrow I = \sqrt{\pi}$. //