Though I tend not to write it out explicitly, we have been doing a lot of substitutions in the integrals we compute. Recall how this goes: given

\[ u \rightarrow g(u) = x \quad \text{C}^1 \text{ function} \]

mapping \([c, d] \rightarrow [a, b] \quad a = g(c), b = g(d)\), in 1-to-1 fashion,

\[ \int_a^b f(x) \, dx = \int_c^d f(g(u)) \, g'(u) \, du. \]

We are after a 2-variable analogue of this.

Suppose

is a 1-to-1 & onto \( C^1 \) map. We'll write

\[ (u, v) \rightarrow \tilde{G}(u, v) = (x(u, v), y(u, v)) \],

and not worry about \( \tilde{G} \) failing to be 1-to-1 on the boundary.
or more generally on a content-two subset. Then for any continuous function \( f: \mathbb{R} \rightarrow \mathbb{R} \),

\[
\int_\mathbb{R} f \, dA = \int \int f(\varphi(u, v)) \left| \det(J_{\varphi}) \right| \, du \, dv
\]

(\star)

where

\[
J_{\varphi} = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\
\frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}
\end{vmatrix}

\]

is the Jacobian or total derivative of \( \varphi \). We now give an informal explanation of how \( \det(J_{\varphi}) \) is obtained.

Begin by enclosing \( T \) in a rectangle \( Q \), and partition \( Q \):

\[
Q_{ij} = (u_0, v_0) \rightarrow (u_0 + \Delta u, v_0 + \Delta v)
\]

\[
\tilde{\varphi}(u_0, v_0) \rightarrow \tilde{\varphi}(u_0 + \Delta u, v_0 + \Delta v)
\]
For simplicity, we assume $\hat{G}$ extends to $\mathbb{R}$. The $\hat{G}(Q_{ij})$ aren't rectangles, but they aren't far from being parallelograms. So we can approximate their area by a cross product:

$$
\mathbf{r}_u := \hat{G}(u_0 + \Delta u, v_0) - \hat{G}(u_0, v_0) \approx J_{\hat{G}}(u_0, v_0) (\Delta u) = \begin{pmatrix}
\frac{\partial \hat{G}(u_0, v_0)}{\partial u} \\
\frac{\partial \hat{G}(u_0, v_0)}{\partial v}
\end{pmatrix} \Delta u
$$

$$
\mathbf{r}_v := \hat{G}(u_0, v_0 + \Delta v) - \hat{G}(u_0, v_0) \approx J_{\hat{G}}(u_0, v_0) (\Delta v) = \begin{pmatrix}
\frac{\partial \hat{G}(u_0, v_0)}{\partial u} \\
\frac{\partial \hat{G}(u_0, v_0)}{\partial v}
\end{pmatrix} \Delta v
$$

$$
a(\hat{G}(Q_{ij})) \approx \left\| \mathbf{r}_u \times \mathbf{r}_v \right\| = 
\begin{vmatrix}
\hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\
\frac{\partial \hat{G}}{\partial u} & \frac{\partial \hat{G}}{\partial v} & 0 \\
\frac{\partial \hat{G}}{\partial v} & \frac{\partial \hat{G}}{\partial u} & 0
\end{vmatrix} \Delta u \Delta v = \left| \det(J_{\hat{G}}(u_0, v_0)) \right| a(Q_{ij}).
$$

So $|J_{\hat{G}}|$ evaluated at a point of $Q_{ij}$ approximates the dilution factor $\frac{a(\hat{G}(Q_{ij}))}{a(Q_{ij})}$, and

$$
\iint_{Q_{ij}} f \, dA = \lim_{\|P\| \to 0} \sum f(\hat{G}(q_{ij})) \frac{a(\hat{G}(Q_{ij}))}{a(Q_{ij})}
$$

as the "norm of the partition", i.e. maximum side-length of a $Q_{ij}$, goes to 0.
\[
\begin{align*}
&= \lim_{\|\Pi\| \to 0} \sum_{i,j} (f \circ \bar{T}) (q_{ij}) \left| \det \left( \bar{T}_{ij} \right) \right| \alpha (Q_{ij}) \\
&= \iint_{\Omega} (f \circ \bar{T}) \left| \det \left( J_{\bar{T}} \right) \right| \, dA.
\end{align*}
\]

A key special case on which we shall now focus is where \((u, v)\) is called \((r, \theta)\) and
\[
\bar{g}(r, \theta) = (r \cos \theta, r \sin \theta).
\]

The Jacobian determinant is
\[
\left| \det \left( J_{\bar{g}} \right) \right| = \left| \begin{array}{cc}
\frac{2}{\partial r} (r \cos \theta) & \frac{2}{\partial \theta} (r \cos \theta) \\
\frac{2}{\partial r} (r \sin \theta) & \frac{2}{\partial \theta} (r \sin \theta)
\end{array} \right| = \left| \begin{array}{cc}
\cos \theta - rsin \theta \\
\sin \theta & \cos \theta
\end{array} \right|
\]
\[
= r \cos \theta (\cos \theta) - (-r \sin \theta) \sin \theta
\]
\[
= r (\cos^2 \theta + \sin^2 \theta) = r,
\]

and so the change-of-variables formula is
\[
\iint_{\Omega} f(x, y) \, dx \, dy = \iint_{\frac{\Omega}{\bar{g}}} (f \circ \bar{g})(r, \theta) \, r \, dr \, d\theta.
\]

Intuitively, this is replacing the Riemann sum
our rectangles by one over angular sectors — which, after all, are the \( G(x, y) \) in this case.

We now turn to several examples of polar integration.

**Ex 1** Find the volume of the solid in the 1st octant bounded by \( z = x^2 + y^2 \), \( x^2 + y^2 = 4 \), and the coordinate planes.

\[
V = \iiint_D (x^2 + y^2) \, dV = \int_0^\infty (r \cos \theta)^2 + (r \sin \theta)^2 \, r \, dr \, d\theta = \int_0^\infty r^2 \, dr \int_0^{\frac{\pi}{2}} \, d\theta
\]

\[
= \frac{1}{3} r^3 \bigg|_0^2 \int_0^{\frac{\pi}{2}} \, d\theta = \frac{2}{3} \cdot 2^3 \int_0^{\frac{\pi}{2}} \, d\theta = \frac{8}{3} \cdot \frac{\pi}{2} = 2\pi.
\]
Ex 2/ Compute the volume of a sphere of radius R. Then find the volume of a cored apple, if the apple is a sphere of radius R and the core is its intersection with a solid cylinder of radius c.

In both cases, you have to integrate the difference between the top and the bottom of the "apple":

\[ V = \int \int \left( \sqrt{R^2-x^2-y^2} - \left( -\sqrt{R^2-x^2-y^2} \right) \right) \, dA \]

\[ = 2 \int_0^R \int_c^R \sqrt{R^2-r^2} \, r \, dr \, d\theta = 4\pi \int_c^R \sqrt{R^2-r^2} \, rdr \]

\[ = 4\pi \left[ -\frac{1}{3} (R^2-r^2)^{3/2} \right]_c^R = \frac{4}{3}\pi (R^2-c^2)^{3/2} \]

\[ = \frac{4}{3}\pi R^3 \quad , \quad h = \text{distance from center to top of cored apple} \]

(h = R if not cored)
Ex 3/ Evaluate \( \int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx \). This is nasty to antidifferentiate in \( x \) or \( y \). Convert it to polar coordinates:

\[
\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2} \, dy \, dx = \int_0^{\pi/2} \int_0^2 r^2 \cos \theta \, dr \, d\theta.
\]

\[
= \frac{8}{3} \int_0^{\pi/2} \cos \theta \, d\theta.
\]

\[
= \frac{8}{3} \left[ \sin \theta \right]_0^{\pi/2} = \frac{8}{3} (1 - \frac{1}{3}) = \frac{16}{9}.
\]

Ex 4/ Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \).

Consider the volume \( V \) under \( e^{-x^2-y^2} \):
we will compute this in two ways.
\( V = \lim_{b \to \infty} \int_{-b}^{b} \int_{-b}^{b} e^{-x^2 - y^2} \, dy \, dx \\
= \lim_{b \to \infty} \left( \int_{-b}^{b} e^{-y^2} \, dy \right) \left( \int_{-b}^{b} e^{-x^2} \, dx \right) \\
= \left( \lim_{b \to \infty} \int_{-b}^{b} e^{-x^2} \, dx \right)^2 = I^2 \)

\( V = \lim_{a \to \infty} \int_{0}^{2\pi} \int_{0}^{a} r \, dr \, d\theta \\
= \lim_{a \to \infty} \frac{1}{2} \int_{0}^{2\pi} \left[ -\frac{1}{2} e^{-r^2} \right]_{r=0}^{r=a} \, d\theta \\
= \lim_{a \to \infty} \frac{1}{2} \int_{0}^{2\pi} (1 - e^{-a^2}) \, d\theta \\
= \lim_{a \to \infty} \pi (1 - e^{-a^2}) = \pi \)

So \( I^2 = \pi \Rightarrow I = \sqrt{\pi} \)