Lecture 41: Green's Theorem in the Plane

The Fundamental Theorem of Calculus tells us how to calculate the integral of $F'(x)$ over an interval $[a, b]$ by using the behavior of $F$ itself on the boundary of the interval:

$$\int_a^b F'(x) \, dx = F(b) - F(a).$$

We had the related result for line integrals of conservative vector fields

$$\oint_C \nabla f \cdot d\mathbf{s} = f(B) - f(A),$$

where $C$ goes from $A$ to $B$. What if we replace $C$ (or $[a, b]$) by something 2-dimensional: is there a similar result?

Green's theorem in the plane states that IF

- $\mathbf{F}(x,y) = (P(x,y), Q(x,y))$ is a $C^1$ vector field on a connected open set $B$
- $C$ is a piecewise smooth, simple closed curve
- $A$ is the region enclosed by $C$
C is oriented so that \( R \) is "on the left" (we say that \( C \) has the counterclockwise orientation).

Then

\[
\oint \mathbf{F} \cdot d\mathbf{r} = \iint_R (Q_x - P_y) \, dA.
\]

**Remarks:**

1. There is a demand implicit in the statement that "the region \( R \) enclosed by \( C \)" lies in \( S \). If \( S \) is not simply connected, then this will not be true for every simple closed curve in \( S \); we can't have \( C \) "go around holes" in \( S \).

2. If \( \mathbf{F} \) is conservative, then we already know that \( \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \). If not, and \( S \) is simply connected, then \( Q_x - P_y \neq 0 \) and Green's Theorem tells us how to use this to calculate \( \oint_C \mathbf{F} \cdot d\mathbf{r} \).

3. Alternatively, we can think of Green as "using the boundary values of \( \mathbf{F} \) on \( \partial R = C \) to compute an integral over the interior, exactly as in the FTC."
Green's Theorem implies the extension of "\( Q_x = P_y \implies \mathbb{F} \) conservative" to all simply connected sets \( S \): in fact, one can define "simply connected"\(^\dagger\) to mean that the region bounded by any simple closed curve in \( S \) is contained in \( S \). Hence the RHS of Green is 0 and so \( \oint_C \mathbb{F} \cdot d\mathbf{r} = 0 \) for all closed loops.

**Ex 1/** Calculate \( \frac{1}{2} \oint_C (-ydx + xdy) \), where \( C \) is a circle of radius \( r \) traversed counterclockwise.

\[ P = -\frac{y}{2}, \quad Q = \frac{x}{2} \implies Q_x - P_y = 1 \]

\[ \implies \frac{1}{2} \oint_C -ydx + xdy = \iint_{\mathbb{R}^2} 1 \, dA = a(R) \]

regardless of what \( R \) is!! Here it's the disk of radius \( R \), and so we get \( \pi R^2 \).

\(^\dagger\) The other definition (I won't prove they are equivalent) is that the complement \( \mathbb{R}^2 \setminus S \) is connected.
Ex 2/ Calculate the line integral

\[ \oint_C (x^3 \sin x - 5y) \, dx + (4x + e^{y^2}) \, dy , \]

where \( C \) is the circle shown.

\[ P = x^3 \sin x - 5y, \quad Q = 4x + e^{y^2} \Rightarrow Q_x - P_y = 4 + 5 = 9 \]

\( \Rightarrow \) \( (\ast) = \iint_R 9 \, dA = 9 \cdot a(R) = 36\pi \). \quad \square

**Warning:** The vector field \( \mathbf{F} = (P, Q) \) must be defined on all of \( R \) for Green to apply. Let \( C \) be the circle in Ex. 2 and consider

\[ \oint_C \left( \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy \right) \]

It's tempting to write \( Q_x - P_y = 0 \) (true in \( R^2 \setminus \{0\} \)) and conclude that \( \int_C \ldots = \iint_R 0 = 0 \). But we know that the answer here is actually \( 2\pi \). The problem is that \( \mathbf{F} \) isn't defined at \((0,0)\), so it's not defined on all of \( R \). The integral \( \frac{1}{2\pi} \oint_C \left( \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy \right) \) is actually the one that computes the "winding number" of \( C \) about \( 0 \) (see Apostol).
So why does Green's Theorem work? For simplicity, assume \( R \) is of type I \& II. Since it's of type I (see proof), we get
\[
\oint_C P \, dx = \int_{e_1} P \, dx + \int_{e_2} P \, dx + \int_{e_3} P \, dx + \int_{e_4} P \, dx
\]

\[
= \int_a^b P(x, g(x)) \, dx + \int_a^b P(x, f(x)) \, dx
\]

\[
= -\int_a^b \{ P(x, f(x)) - P(x, g(x)) \} \, dx
\]

\[
= -\int_a^b \int_{g(x)}^{f(x)} P_y(x, y) \, dy \, dx = -\iint_R P_y \, dA.
\]

By a "symmetric" argument (swapping roles of \( x \& y \)),
\( R \) of type II \( \Rightarrow \oint_C Q \, dy = \iint_R Q_x \, dA. \)
(The sign changes by swapping \( x \& y \) changes the orientation of \( C \) to a clockwise one)

\[
\Rightarrow \oint_C P \, dx + Q \, dy = \iint_R (Q_x - P_y) \, dA.
\]

If \( R \) isn't both of type I \& type II, we can chop it into a finite union of subregions which are
Adding $\sum \iiint_{R_i} \mathbf{F} \cdot d\mathbf{A}$ gives $\iiint_{R} \mathbf{F} \cdot d\mathbf{A}$ on the RHS (of Green). On the LHS, $\sum \overrightarrow{\partial R_i} = \overrightarrow{\partial R} = C \Rightarrow \sum \oint_{\partial R_i} \mathbf{F} = \oint_{C} \mathbf{F}$ because of the cancellations along the path components in red. \hfill \square

In fact, Green's Theorem even applies to a region $R$ with one or more holes ("multiply connected" region), provided that each part of the boundary is oriented so that $R$ remains on the left.

In the picture shown, we write $\partial[R] = C_1 + C_2 + C_3$.

Thinking back to $\mathbf{F} = \left( -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$, consider an annulus $R$ with boundary $C_1 - C_2$, where $C_r$ means the counterclockwise circle of radius $r$. Now $\mathbf{F}$ is defined on $R$ (since $R$ avoids the origin), and $Q_x - P_y = 0$ on $R$. So by Green's Theorem

$$0 = \iint_{R} (Q_x - P_y) \, dA = \iint_{C_1} \mathbf{F} \cdot d\mathbf{A} - \iint_{C_2} \mathbf{F} \cdot d\mathbf{A}$$

(from Lecture 36)
\[ \oint_{C_n} \mathbf{F} \cdot d\mathbf{r} = 2\pi \] for any radius \( n \). Now this isn't
hard to check by computation, but suppose we
replace \( C_n \) by "any curve \( C' \) that goes once
around the origin" — we still get \( 2\pi \)
by this argument!

**Ex 3**/ Find the area under one arch
of the cycloid parameterized by
\( \mathbf{r}(t) = (t - \sin t, 1 - \cos t) \).

\[
A(R) = \iint_{R} 1 \, dA = \frac{1}{2} \oint_{C_1} (-y \, dx + x \, dy)
= \frac{1}{2} \oint_{C_1} \mathbf{e}_1 \cdot (\mathbf{r}'(t) \times \mathbf{r}'(t)) \, dt
\]
\[
= \frac{1}{2} \oint_{C_1} (-\{ \cos t - 1 \} \{ 1 - \cos t \} + \{ t - \sin t \} \{ \sin t \}) \, dt
\]
\[
= \oint_{C_1} \{ 1 - \cos t - \frac{1}{2} t \sin t \} \, dt
\]
\[
= \left[ t - \sin t + \frac{1}{2} t \cos t - \frac{1}{2} \sin t \right]_0^{2\pi}
\]
\[
= 3\pi.
\]

Fun problem: apply to finding areas of polygons...
Finally, we turn to some vector forms of Green's Theorem. The first is a direct translation, using $dr^2 = \hat{T} \, ds$ and writing

$$\text{Curl} \, \vec{F} := (Q_x - P_y) \hat{k},$$

so that Green's Theorem becomes

$$\oint_{\partial R} \vec{F} \cdot \hat{T} \, ds = \iint_R (\text{Curl} \, \vec{F}) \cdot \hat{k} \, dA.$$

This is Stokes's Theorem in the plane, and says that the total circulation of $\vec{F}$ around the boundary equals the integral of a measure of circulation over the interior.

Next recall that if $C$ is parametrized by arclength, then the unit tangent and normal vectors are given by

$$\hat{T}(s) = \left( x'(s), y'(s) \right)$$
$$\hat{N}(s) = \left( y'(s), -x'(s) \right).$$

Write

$$\oint_C \vec{F} \cdot \hat{N} \, ds = \oint_C (P_x(x(s), y(s)), y'(s), -x'(s)) \, ds = \int_C Q_x \, dx + P_y \, dy = \iint_R (P_x - (-Q_y)) \, dA \quad \text{Green}$$

$$= \iint_R (P_x + Q_y) \, dA.$$
Defining the divergence

\[ \text{div}(\vec{F}) = P_x + Q_y , \]

we have proved Gauss’s Divergence Theorem in the plane:

\[ \oint_{\Gamma} \vec{F} \cdot \hat{n} \, ds = \iint_{R} \text{div}(\vec{F}) \, dA . \]

Here \( \vec{F} \cdot \hat{n} \) is the normal component of \( \vec{F}(\xi, \eta) \). If \( \vec{F} \) is a fluid velocity field, then this is the flow (outward flow through the boundary) per unit length; and the left-hand integral is therefore the total flux of \( \vec{F} \) across the boundary \( C = \partial[R] \).

Ex 4: If \( \vec{F} = (x^2 + y^2, 2xy) \), find the flux of \( \vec{F} \) across the boundary of the unit square and the circulation of \( \vec{F} \) around the boundary.

Flux = \[ \oint_{\partial[R]} \vec{F} \cdot \hat{n} \, ds = \iint_{R} \left( P_x + Q_y \right) \, dA = \int_0^1 \int_0^1 (2x^2 + 2x) \, dy \, dx\]

= \[ \int_0^1 4x \, dx = 2 \]

Circulation = \[ \oint_{\partial[R]} \vec{F} \cdot \vec{T} \, ds = \iint_{R} \left( Q_x - P_y \right) \, dA = \int_0^1 \int_0^1 (2y - 2y) \, dy \, dx \]

= 0.