Lecture 41: Green's Theorem in the Plane

The Fundamental Theorem of Calculus tells us how to calculate the integral of $F'(x)$ over an interval $[a,b]$ by using the behavior of F itself on the boundary of the interval:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

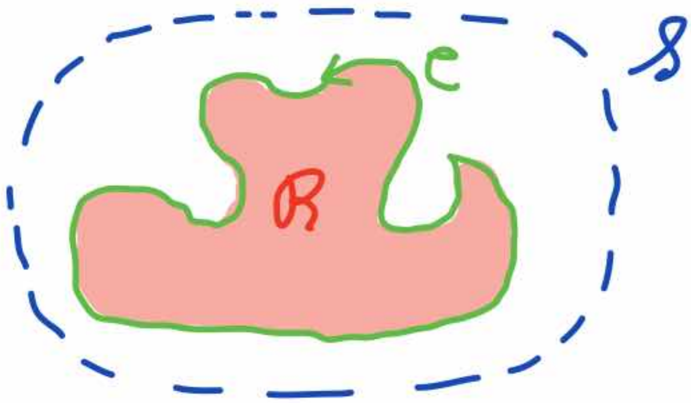
We had the related result for line integrals of conservative vector fields

$$\int_{\mathcal{C}} \vec{\nabla} f \cdot d\vec{r} = f(B) - f(A),$$

where $\mathcal{C}$ goes from A to B. What if we replace $\mathcal{C}$ (or $[a,b]$) by something 2-dimensional: is there a similar result?

Green's theorem in the plane states that IF

- $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is a C^1 vector field on a connected open set $\mathcal{S}$
- $\mathcal{C}$ is a piecewise smooth, simple closed curve
- $\mathcal{R}$ is the region enclosed by $\mathcal{C}$



- C is oriented so that R is "on the left" (we say that C has the counterclockwise orientation).

THEN

the loop integral notation is used for line integrals on simple closed curves.

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{i.e. } \oint_C P\,dx + Q\,dy} = \iint_R (Q_x - P_y)\,dA.$$

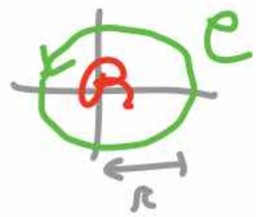
Remarks: ① There is a demand implicit in the statement that "the region R enclosed by C" lies in $\mathcal{S}$. If $\mathcal{S}$ is not simply connected, then this will not be true for every simple closed curve in $\mathcal{S}$: we can't have C "go around holes" in $\mathcal{S}$.

② If $\vec{F}$ is conservative, then we already know that $\oint_C \vec{F} \cdot d\vec{r} = 0$. If not, and $\mathcal{S}$ is simply connected, then $Q_x - P_y \neq 0$ and Green's Theorem tells us how to use this to calculate $\oint_C \vec{F} \cdot d\vec{r}$.

③ Alternatively, we can think of Green as "using the boundary values of $\vec{F}$" on $\partial R (= C)$ to compute an integral over the interior, exactly as in the FTC.

(4) Green's Theorem implies the extension of "$Q_x = P_y \Rightarrow \vec{F}$ conservative" to all simply connected sets $\mathcal{S}$: in fact, one can define "simply connected"† to mean that the region bounded by any simple closed curve in $\mathcal{S}$ is contained in $\mathcal{S}$. Hence the RHS of Green is 0 and so $\oint_{\mathcal{C}} \vec{F}\cdot d\vec{r} = 0$ for all closed loops.

Ex 1/ Calculate $\frac{1}{2}\oint_{\mathcal{C}}(-y\,dx + x\,dy)$, where $\mathcal{C}$ is a circle of radius r traversed counterclockwise.



$$P = \frac{-y}{2},\ Q = \frac{x}{2} \Rightarrow Q_x - P_y = 1$$

$$\underset{\text{Green}}{\Longrightarrow} \frac{1}{2}\oint_{\mathcal{C}} -y\,dx + x\,dy = \iint_{R} 1\, dA = a(R),$$

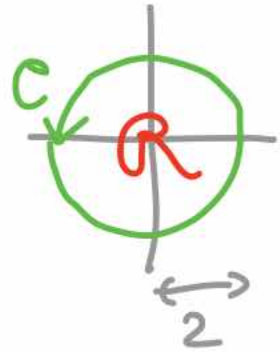
regardless of what R is!! Here it's the disk of radius r, and so we get πr^2. //

† The other definition (I won't prove they are equivalent) is that the complement $\mathbb{R}^2 \setminus \mathcal{S}$ is connected.

Ex 2/ Calculate the line integral

$$(*) \quad \oint_{\mathcal{C}} (x^3 \sin x - 5y)\,dx + (4x + e^{y^2})\,dy\,,$$

where $\mathcal{C}$ is the circle shown.



$$P = x^3 \sin x - 5y\,,\quad Q = 4x + e^{y^2} \Rightarrow Q_x - P_y = 4 + 5 = 9$$

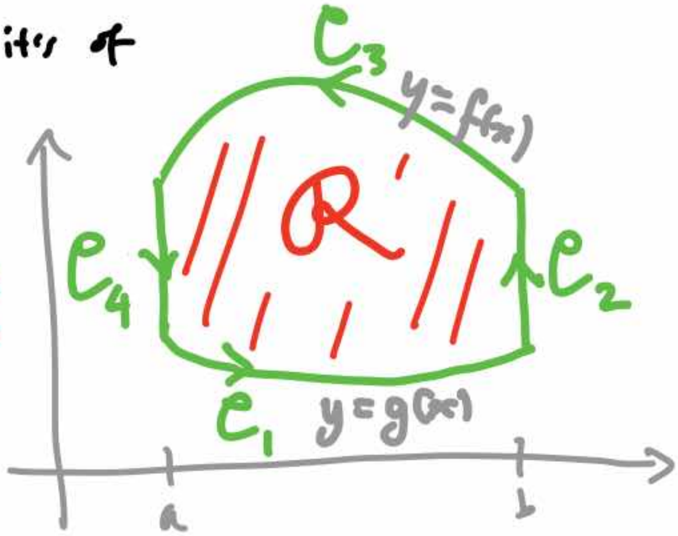
$$\Rightarrow (*) = \iint_{\mathcal{R}} 9\,dA = 9 \cdot a(\mathcal{R}) = 36\pi\,. \qquad //$$

WARNING: The vector field $\vec{F} = (P, Q)$ must be defined on all of $\mathcal{R}$ for Green to apply. Let $\mathcal{C}$ be the circle in Ex. 2 and consider

$$\oint_{\mathcal{C}} \underbrace{\frac{-y}{x^2+y^2}}_{P}\,dx + \underbrace{\frac{x}{x^2+y^2}}_{Q}\,dy$$

It's tempting to write $Q_x - P_y = 0$ (true on $\mathbb{R}^2 \setminus \{\vec{0}\}$) and conclude that $\oint_{\mathcal{C}} \cdots = \iint_{\mathcal{R}} 0 = 0$. But we know that the answer here is actually 2π. The problem is that $\vec{F}$ isn't defined at $(0,0)$, so it's not defined on all of $\mathcal{R}$. The integral $\frac{1}{2\pi} \oint_{\mathcal{C}} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$ is actually the one that computes the "winding number" of $\mathcal{C}$ about $\vec{0}$ (see Apostol).

So why does Green's Theorem work? For simplicity, assume $\mathcal{R}$ is of type I & II. Since it's of type I (see picture), we get



$$\oint_C P\,dx = \int_{C_1} P\,dx + \underbrace{\cancel{\int_{C_2} P\,dx}}_{0} + \int_{C_3} P\,dx + \underbrace{\cancel{\int_{C_4} P\,dx}}_{0}$$

since x is constant on C_2, C_4

$$= \int_a^b P(x,g(x))\,dx + \int_b^a P(x,f(x))\,dx$$

$$= -\int_a^b \{P(x,f(x)) - P(x,g(x))\}\,dx$$

$$= -\int_a^b \int_{g(x)}^{f(x)} P_y(x,y)\,dy\,dx = -\iint_{\mathcal{R}} P_y\,dA.$$

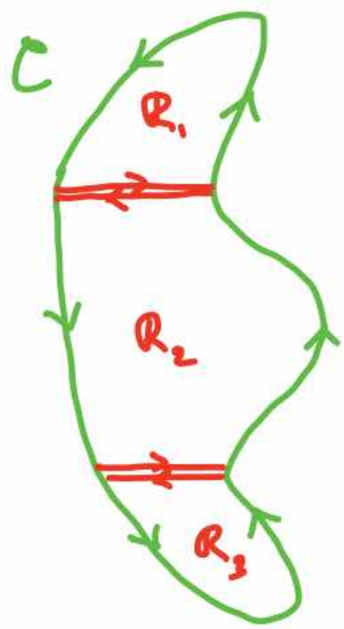
By a "symmetric" argument (swapping roles of x & y),

$\mathcal{R}$ of type II $\Rightarrow$ $\oint_C Q\,dy = \iint_{\mathcal{R}} Q_x\,dA$.

(The sign changes b/c swapping x & y changes the orientation of C to a clockwise one)

$$\Longrightarrow \oint_C P\,dx + Q\,dy = \iint_{\mathcal{R}} (Q_x - P_y)\,dA.$$

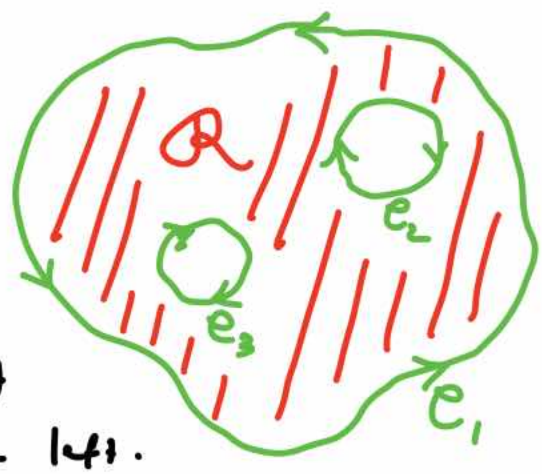
If $\mathcal{R}$ isn't both of type I & type II, we can chop it into a finite union of subregions which <u>are</u>:



Adding $\sum_i \iint_{R_i} \cdots \, dA$ gives $\iint_R \cdots \, dA$ on the RHS (of Green). On the LHS,

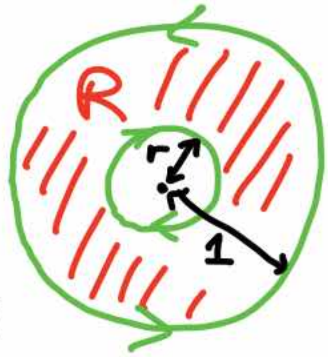
$$\sum_i \partial[R_i] = \partial[R] = C \Rightarrow \sum_i \oint_{\partial[R_i]} = \oint_C$$

because of the cancellations along the path components in red. $\square$



In fact, Green's Theorem even applies to a region R with one or more holes ("multiply connected" region), provided that each part of the boundary is oriented so that R remains on the left. In the picture shown, we write $\partial[R] = C_1 + C_2 + C_3$.

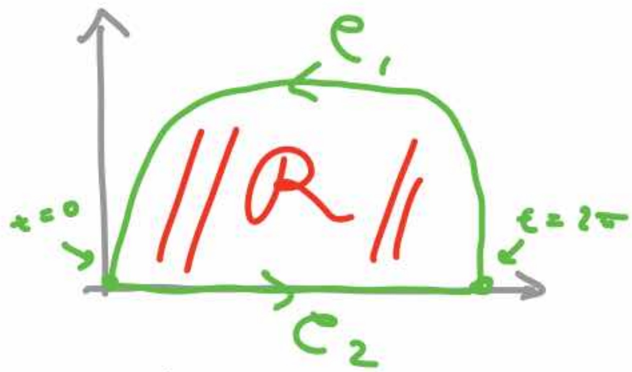
Thinking back to $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$, consider an annulus R with boundary $C_1 - C_r$, where C_r means the counterclockwise circle of radius r. Now $\vec{F}$ is defined on R (since R avoids the origin), and $Q_x - P_y = 0$ on R. So by Green's Theorem



$$0 = \iint_R (Q_x - P_y)\, dA = \oint_{\partial[R]} \vec{F}\cdot d\vec{r} = \overbrace{\oint_{C_1} \vec{F}\cdot d\vec{r}}^{=2\pi \text{ (from Lecture 36)}} - \oint_{C_r} \vec{F}\cdot d\vec{r}$$

$\Rightarrow \oint_{C_R} \vec{F} \cdot d\vec{r} = 2\pi$ for any radius R. Now this isn't hard to check by computation, but suppose we replace C_R by "any curve C' that goes once around the origin" — we still get 2π by this argument!

Ex 3/ Find the area under one arch of the cycloid parametrized by $\vec{r}(t) = (t - \sin t, 1 - \cos t)$.



$$a(R) = \iint_R 1\, dA = \frac{1}{2} \oint_{C_1 + \cancel{C_2}^{\,0}\text{(why?)}} -y\,dx + x\,dy = \frac{1}{2}\int_{2\pi}^{0} \{-y(t)x'(t) + x(t)y'(t)\}\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} -\{(\cos t - 1)(1-\cos t) + (t - \sin t)(\sin t)\}\,dt$$

$$= \int_0^{2\pi} \{1 - \cos t - \tfrac{1}{2} t \sin t\}\,dt$$

$$= \left[t - \sin t + \tfrac{1}{2} t \cos t - \tfrac{1}{2}\sin t\right]_0^{2\pi}$$

$$= 3\pi.$$

//

Fun problem: apply to finding areas of polygons...

Finally, we turn to some vector forms of Green's Theorem.

The first is a direct translation, using $d\vec{r} = \hat{T}\,ds$ and writing

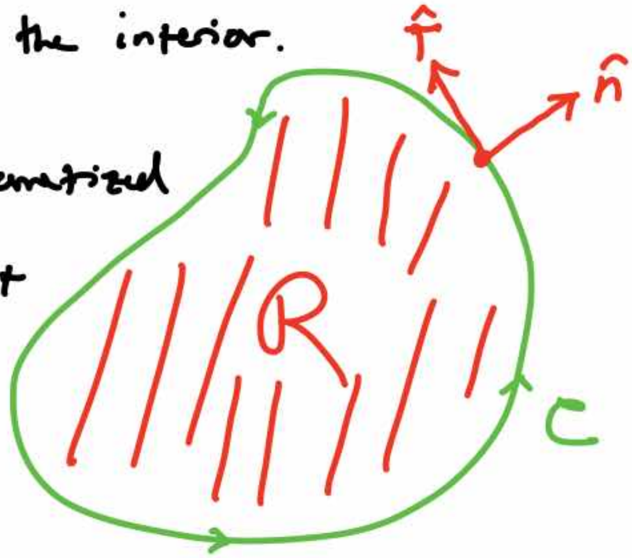
$$\operatorname{curl}\vec{F} := (Q_x - P_y)\,\hat{k},$$

recall $\hat{i}, \hat{j}, \hat{k}$ basis of $\mathbb{R}^3$

So that Green's Theorem becomes

$$\oint_{\partial[R]} \vec{F}\cdot\hat{T}\,ds = \iint_R (\operatorname{curl}\vec{F})\cdot\hat{k}\,dA .$$

This is <u>Stokes's Theorem in the plane</u>, and says that the total <u>circulation</u> of $\vec{F}$ around the boundary equals the integral of a measure of circulation over the interior.



Next recall that if C is parametrized by arclength, then the unit tangent and normal vectors are given by

$\hat{T}(s) = (x'(s), y'(s))$

$\hat{n}(s) = (y'(s), -x'(s))$. Write

$$\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C (P(\vec{r}(s)), Q(\vec{r}(s)))\cdot(y'(s), -x'(s))\,ds = \oint_C -Q\,x'(s)\,ds + P\,y'(s)\,ds$$

$$= \oint_C -Q\,dx + P\,dy \underset{\text{Green}}{=\!=} \iint_R (P_x - (-Q_y))\,dA$$

$$= \iint_R (P_x + Q_y)\,dA .$$

Defining the divergence

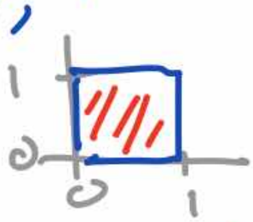
$$\mathrm{div}(\vec{F}) = P_x + Q_y ,$$

we have proved Gauss's Divergence Theorem in the plane:

$$\oint_{\partial[R]} \vec{F}\cdot\hat{n}\, ds = \iint_R \mathrm{div}(\vec{F})\, dA .$$

Here $\vec{F}\cdot\hat{n}$ is the normal component of $\vec{F}(\vec{r}(t))$. If $\vec{F}$ is a fluid velocity field, then this is the flux (outward flow through the boundary) per unit length; and the left-hand integral is therefore the total flux of $\vec{F}$ across the boundary $\mathcal{C} = \partial[R]$.

Ex 4/ If $\vec{F} = (x^2+y^2, 2xy)$, find the flux of $\vec{F}$ across the boundary of the unit square



, and the circulation of $\vec{F}$ around the boundary.

$$\text{flux} = \oint_{\partial[R]} \vec{F}\cdot\hat{n}\, ds \underset{\text{Gauss}}{=} \iint_R \overbrace{(P_x+Q_y)}^{\text{integrate } \mathrm{div}(\vec{F})} dA = \int_0^1\int_0^1 (2x+2x)\, dy\, dx$$

$$= \int_0^1 4x\, dx = 2$$

$$\text{circulation} = \oint_{\partial[R]} \vec{F}\cdot\hat{T}\, ds \underset{\text{Stokes}}{=} \iint_R \overbrace{(Q_x-P_y)}^{\text{integrate } \mathrm{curl}(\vec{F})\cdot\hat{k}} dA = \int_0^1\int_0^1 (2y-2y)\, dy\, dx$$

$$= 0 .$$

//