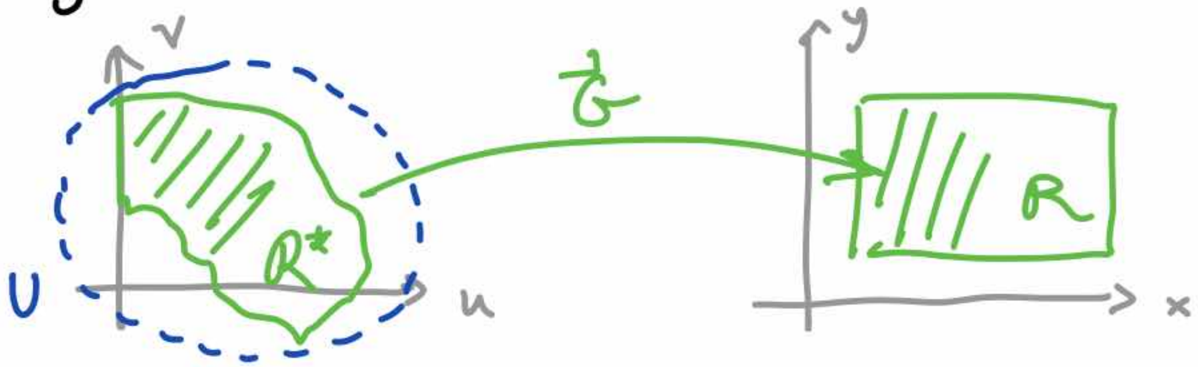
Lecture 42: Change of Variable (II)

In Lecture 40, we gave a heuristic derivation of the change-of-variable formula which rested on the interpretation of the Jacobian determinant as the local expansion factor for area.

Today I want to use Green's Theorem to give a proof, starting with the case where $\mathcal{S}$ is a rectangle $\mathcal{R}$.



Let $\vec{G}: U \to \mathbb{R}^2$ be a C^2 vector-valued function defined on an open set of $\mathbb{R}^2$. Let $\mathcal{R}^* = \vec{G}^{-1}(\mathcal{R})$ be the inverse image of a closed rectangle $\mathcal{R}$; and assume that the restriction of $\vec{G}$ to $\mathcal{R}^*$ is 1-to-1 and onto, with $\det(J_{\vec{G}}) > 0$. Writing $Q = x$ and $P = 0$, Green's Theorem for $\mathcal{R}$ says that

$$\iint_{\mathcal{R}} 1\,dx\,dy = \iint_{\mathcal{R}} (Q_x - P_y)\,dx\,dy = \oint_{\partial[\mathcal{R}]} P\,dx + Q\,dy = \underbrace{\oint_{\partial[\mathcal{R}]} x\,dy}_{(\mathrm{I})},$$

while on $\mathcal{R}^*$ gives

$$\iint_{R^*} (\det J_{\vec{G}})\,du\,dv = \iint_{R^*} (x_u y_v - x_v y_u)\,du\,dv$$

add for free

$$= \iint_{R^*} \Big\{ \underbrace{(x_u y_v + x y_{vu})}_{(x y_v)_u} - \underbrace{(x y_{vu} + x_v y_u)}_{(x y_u)_v} \Big\}\,du\,dv \qquad (x y_v)_u - (x y_u)_v = Q_u - P_v$$

$$\begin{cases} P = x y_u \\ Q = x y_v \end{cases}$$

$$= \oint_{\partial[R^*]} P\,du + Q\,dv$$

$$= \underbrace{\oint_{\partial[R^*]} x y_u\,du + x y_v\,dv}_{(\mathrm{II})}$$

Parametrizing $\partial[R^*]$ by

$$\vec{r}(t) = (u(t), v(t)) \quad \text{on} \quad a \le t \le b$$

and $\partial[R]$ by

$$(\vec{G} \circ \vec{r})(t) = (\overbrace{x(u(t), v(t))}^{=: x(t)}, \overbrace{y(u(t), v(t))}^{=: y(t)}),$$

we have

$$(\mathrm{I}) \quad \oint_{\partial[R]} x\,dy \overset{\text{using } \vec{G}\circ\vec{r}}{=} \int_a^b x(t)\, y'(t)\,dt$$

by the Chain rule ("y_u" means $\frac{\partial y}{\partial u}(u(t), v(t))$)

$$= \int_a^b x(t) \big(y_u u'(t) + y_v v'(t) \big)\,dt$$

$$= \int_a^b x y_u u'(t)\,dt + x y_v v'(t)\,dt$$

using $\vec{r}$

$$= \oint_{\partial[R^*]} x y_u\,du + x y_v\,dv. \quad (\mathrm{II})$$

So $(\mathrm{I}) = (\mathrm{II})$ and $\boxed{\iint_R dx\,dy = \iint_{R^*} (\det J_{\vec{G}})\,du\,dv.}$ $(*)$

[N.B.: The assumptions that $\vec{G}$ is a bijection from $R^* \to R$, together with $\det J_{\vec{G}} > 0$, ensure that if $\vec{r}$ traverses $\partial[R^*]$ once counterclockwise then $\vec{G}\circ\vec{r}$ traverses $\partial[R]$ once counterclockwise (I won't prove this).]

Next suppose that $s: R \to \mathbb{R}$ is a step function, taking the value c_{ij} on R_{ij}. We have by (*) on each R_{ij}

$$a(R_{ij}) = \iint_{R_{ij}} dx\,dy = \iint_{\underset{\vec{G}^{-1}(R_{ij})}{\underset{\shortparallel}{R^*_{ij}}}} \det J_{\vec{G}}\,du\,dv$$

hence

$$\boxed{\iint_R s\,dx\,dy} = \sum_{i,j} c_{ij}\,a(R_{ij}) = \sum_{i,j} c_{ij}\iint_{R^*_{ij}} \det J_{\vec{G}}\,du\,dv$$

since $s\circ\vec{G}$ takes the value c_{ij} on R^*_{ij} →

$$= \sum_{i,j}\iint_{R^*_{ij}} (s\circ\vec{G})\det J_{\vec{G}}\,du\,dv$$

since R^* is made up of the R^*_{ij} →

$$= \boxed{\iint_{R^*} (s\circ\vec{G})\det J_{\vec{G}}\,du\,dv.} \qquad (**)$$

So we have the change-of-variable formula for step functions.

Finally suppose $f: R \to \mathbb{R}$ is integrable, and $s \le f \le t$ for step functions s, t. Then $s\circ\vec{G} \le f\circ\vec{G} \le t\circ\vec{G} \xRightarrow{\det(J_{\vec{G}})>0}$

$$\iint_{R^*}(s\circ\vec{G})\det J_{\vec{G}}\,du\,dv \le \iint_{R^*}(f\circ\vec{G})\det J_{\vec{G}}\,du\,dv \le \iint_{R^*}(t\circ\vec{G})\det J_{\vec{G}}\,du\,dv$$

$$\overset{(**)}{=}\ \iint_R s\,dx\,dy \qquad\qquad \overset{(**)}{=}\ \iint_R t\,dx\,dy$$

Taking sup on LHS ($=\underline{I}(f)$) and inf on RHS ($=\overline{I}(f)$) gives

$$\underline{I}(f) \le \iint_{R^*}(f\circ\vec{G})\det J_{\vec{G}}\,du\,dv \le \overline{I}(f)$$

Since f is integrable, both of these, hence the middle, are equal to $\iint_R f\,dx\,dy$.

So for any integrable function we now have rigorously proved the change-of-variable formula

$$\boxed{\iint_R f\,dx\,dy = \iint_{R^*} (f\circ\vec{G})\det J_{\vec{G}}\,du\,dv.} \qquad (\dagger)$$

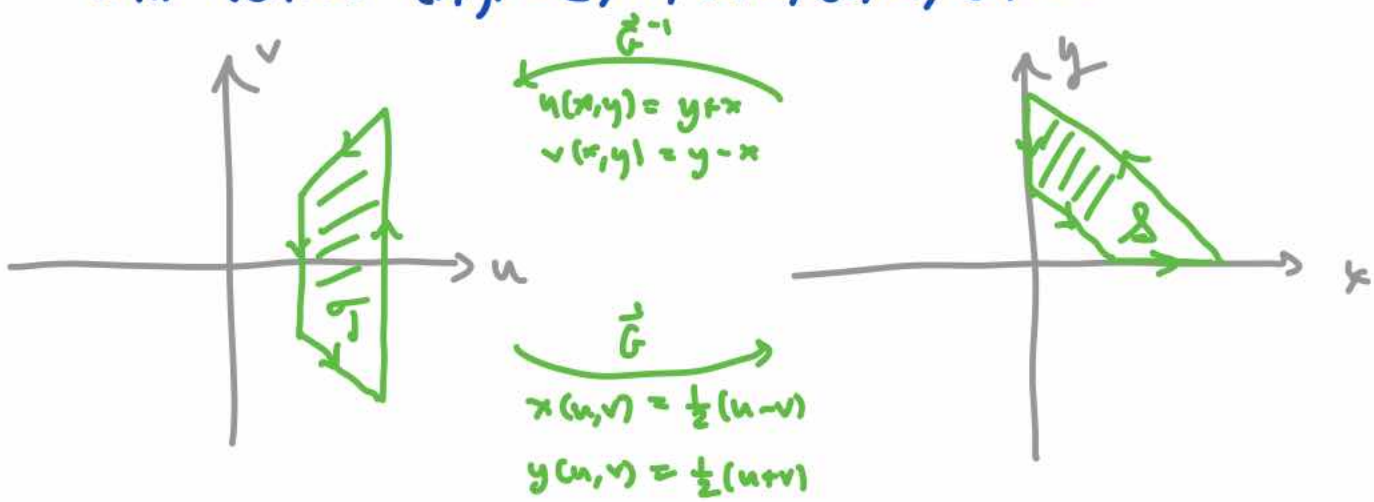
... But that was with $\mathcal{S} = R$ a rectangle. For a more general $\mathcal{S}$, let $h : \mathcal{S} \to \mathbb{R}$ be the function we need to integrate and $f = \tilde{h} : R \to \mathbb{R}$ its extension by zero to a rectangle enclosing $\mathcal{S}$. Writing $\mathcal{T} = \vec{G}^{-1}(\mathcal{S}) \subset R^*$, $f\circ\vec{G}$ is clearly the extension of $h\circ\vec{G} : \mathcal{T} \to \mathbb{R}$ by zero to R^*. So we get by $(\dagger)$

$$\iint_{\mathcal{S}} h\,dx\,dy = \iint_{\mathcal{T}} (h\circ\vec{G})\det J_{\vec{G}}\,du\,dv,$$

the general change-of-variable formula for $\vec{G} : \mathcal{T} \to \mathcal{S}$. $\square$

Let's do some non-polar-coordinate examples.

Ex 1/ Compute $\iint_{\mathcal{S}} \cos\left(\frac{\overbrace{y-x}^{v}}{\underbrace{y+x}_{u}}\right) dA$, where $\mathcal{S}$ is the quadrilateral with vertices $(x,y) = (1,0), (2,0), (0,2), (0,1)$.



$$J_{\vec{G}} = \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \Rightarrow \det J_{\vec{G}} = \frac{1}{2}$$

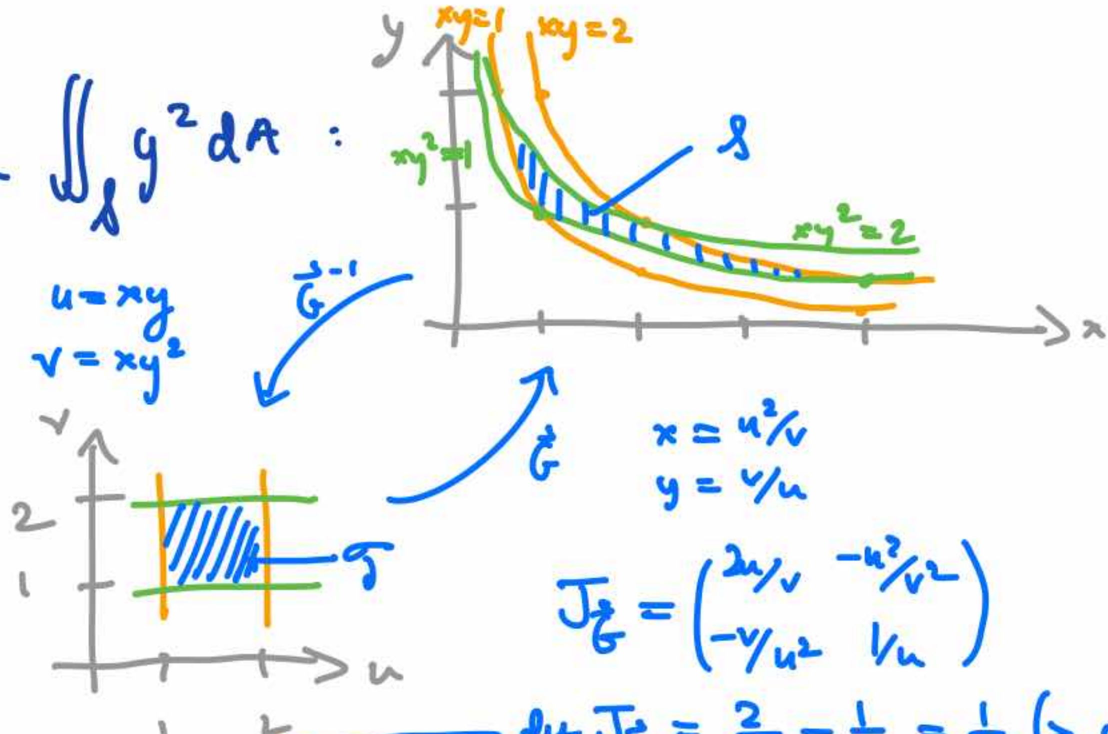
$$\Rightarrow \iint_{\mathcal{S}} \cos\left(\frac{y-x}{y+x}\right) dx\,dy = \iint_{\mathcal{T}} \cos\left(\frac{v}{u}\right) \frac{1}{2}\, du\,dv$$

$$= \frac{1}{2}\int_1^2 \int_{-u}^{u} \cos\left(\frac{v}{u}\right) dv\,du = \frac{1}{2}\int_1^2 \left[u \cdot \sin\left(\frac{v}{u}\right)\right]_{v=-u}^{v=u} du$$

$$= \frac{1}{2}\int_1^2 \{u \sin(1) - u\sin(-1)\}\, du = \sin(1)\int_1^2 u\, du$$

$$= \frac{3}{2}\sin(1).$$

Ex 2/ Calculate $\iint_{\mathcal{S}} y^2\, dA$:



$$J_{\vec{G}} = \begin{pmatrix} 2u/v & -u^2/v^2 \\ -v/u^2 & 1/u \end{pmatrix}$$

$$\det J_{\vec{G}} = \frac{2}{v} - \frac{1}{v} = \frac{1}{v} \ (>0)$$

$$\iint_{\mathcal{S}} y^2\, dx\,dy = \iint_{\mathcal{T}} \left(\frac{v}{u}\right)^2 \frac{1}{v}\, du\,dv$$

$$= \int_1^2\int_1^2 \frac{v}{u^2}\, du\,dv = \left(\int_1^2 v\, dv\right)\left(\int_1^2 \frac{1}{u^2}\, du\right)$$

$$= \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = 3/4.$$