Lecture 43: Multiple Integrals

Let $\mathcal{S} \subset \mathbb{R}^n$ be a region (= closed, bounded, connected subset)

$f : \mathcal{S} \to \mathbb{R}$ a function.

Define $\int_{\mathcal{S}} f \, dV \left(= \int \cdots \int_{\mathcal{S}} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n \right)$ as before:

also written $d\underline{x}$, $d^n x$

begin with a step function s on a rectangle $R = [\vec{a}, \vec{b}]$,

subdivided into $R_{\underline{i} = (i_1, \dots, i_n)}$ $(1 \le i_1 \le n, \dots, 1 \le i_n \le m)$ on which

$s|_{R_{\underline{i}}} \equiv c_{\underline{i}}$ (= constants), and set

n-volume of the box $R_{\underline{i}}$

$$\int_R s \, dV := \sum_{\underline{i}} c_{\underline{i}} V(R_{\underline{i}}).$$

Taking R to contain $\mathcal{S}$, we extend f to

$$\tilde{f}(\vec{x}) := \begin{cases} f(\vec{x}), & \vec{x} \in \mathcal{S} \\ 0, & \vec{x} \in R \setminus \mathcal{S} \end{cases} \quad \text{on } R,$$

and declare

$$f \text{ integrable (on } \mathcal{S}) \overset{\text{def.}}{\iff} \tilde{f} \text{ integrable (on } R).$$

$$\overset{\text{def.}}{\iff} \sup\left\{ \int_R s \, dV \;\middle|\; s \le \tilde{f} \text{ step} \right\} = \inf\left\{ \int_R t \, dV \;\middle|\; t \ge \tilde{f} \text{ step} \right\}$$

in which case this defines $\int_{\mathcal{S}} f \, dV$

Theorem 1: If $\partial \mathcal{S}$ has content zero (e.g. anything that can be described as a graph, like $x_n = g(x_1, \dots, x_{n-1})$ w/ g continuous) and f is continuous, then f is integrable.

Theorem 2 (FUBINI): $\displaystyle\int_{[\vec{a},\vec{b}]} \tilde{f}\, dV = \int_{a_n}^{b_n} \cdots \left(\int_{a_2}^{b_2} \left(\int_{a_1}^{b_1} \tilde{f}(x_1, \dots, x_n)\, dx_1 \right) dx_2 \right) \cdots dx_n.$

(or in any other order)

Corollary: If (for ϕ_1, ϕ_2 continuous functions on $\mathcal{D} \subset \mathbb{R}^{n-1}$)

$$\mathcal{S} = \left\{ (x_1, \dots, x_n) \mid (x_1, \dots, x_{n-1}) \in \mathcal{D},\ x_n \in [\phi_1(x_1, \dots, x_{n-1}), \phi_2(x_1, \dots, x_{n-1})] \right\},$$

then $\displaystyle\int_{\mathcal{S}} f\, dV = \int_{\mathcal{D}} \left(\int_{\phi_1(x_1, \dots, x_{n-1})}^{\phi_2(x_1, \dots, x_{n-1})} f(x_1, \dots, x_n)\, dx_n \right) dx_1 \cdots dx_{n-1}.$

Theorem 3: If $\vec{G} : \mathcal{T} \to \mathcal{S}$ is a 1-to-1, onto, C^1 mapping,

$\vec{G}(\overbrace{u_1, \dots, u_n}^{\vec{u}}) = (x_1(\vec{u}), \dots, x_n(\vec{u}))$, then (writing $J_{\vec{G}} = [\partial x_i / \partial u_j]$)

$$\int_{\mathcal{S}} f(\vec{x})\, d\underline{x} = \int_{\mathcal{T}} f(\vec{G}(\vec{u})) \left| \det J_{\vec{G}}(\vec{u}) \right| d\underline{u}.$$

Remark: If a small rectangular box $R_{\underline{i}} \subset \mathcal{T}$ has volume $\Delta u_1 \cdots \Delta u_n$ and vertex at $\vec{u} = \vec{a}$, then $\vec{G}(R_{\underline{i}})$ is approximated by a parallelepiped of volume $|\det J_{\vec{G}}(\vec{a})| \Delta u_1 \cdots \Delta u_n$. If $n = 3$ this volume is the scalar triple-product of $J_{\vec{G}}(\vec{a}) \begin{pmatrix} \Delta u_1 \\ 0 \\ 0 \end{pmatrix}$, $J_{\vec{G}}(\vec{a}) \begin{pmatrix} 0 \\ \Delta u_2 \\ 0 \end{pmatrix}$, and $J_{\vec{G}}(\vec{a}) \begin{pmatrix} 0 \\ 0 \\ \Delta u_3 \end{pmatrix}$.

Ex 0/ $\displaystyle V([\vec{a},\vec{b}]) = \int_{[\vec{a},\vec{b}]} 1\, dV = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} 1\, dx_n \cdots dx_1$

$\displaystyle = \left(\int_{a_1}^{b_1} 1\, dx_1 \right) \cdots \left(\int_{a_n}^{b_n} 1\, dx_n \right) = \prod_{i=1}^{n} (b_i - a_i).$ //

Ex 1/ Calculate $\int_{[0,1]^n} \frac{1}{1-x_1\cdots x_n}\, d\underline{x}$.

Fubini

$$= \int_0^1 \cdots \left(\int_0^1 \frac{1}{1-x_1\cdots x_n}\, dx_1 \right) \cdots dx_n$$

$$= \int_0^1 \cdots \left(\int_0^1 \sum_{k=0}^{\infty} x_1^k \cdots x_n^k \, dx_1 \right) \cdots dx_n$$

$$= \sum_{k=0}^{\infty} \int_0^1 \cdots \left(\int_0^1 x_1^k \cdots x_n^k \, dx_1 \right) \cdots dx_n$$

uniform convergence doesn't quite hold, and an argument is required

$$= \sum_{k=0}^{\infty} \left(\int_0^1 x^k \, dx \right)^n = \sum_{k=0}^{\infty} \frac{1}{(k+1)^n}$$

$$= \sum_{l=1}^{\infty} \frac{1}{l^n} =: \zeta(n)$$ ← called "zeta values".

The even ones are powers of π times rational numbers; the odd ones are still poorly understood even though people have studied them for hundreds of years!

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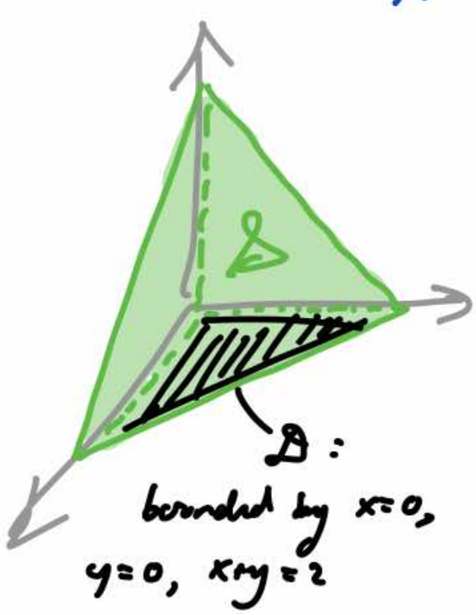
Ex 2/ Calculate $\iiint_S y \, dV$, where

S is the tetrahedron bounded by

$x=0$, $y=0$, $z=0$, and $x+y+z=2$:



$$\iiint_S y \, dV = \iint_D \left(\int_0^{2-x-y} y \, dz \right) dy\, dx$$

$$= \iint_D y\,(2 - x - y)\, dy\, dx$$

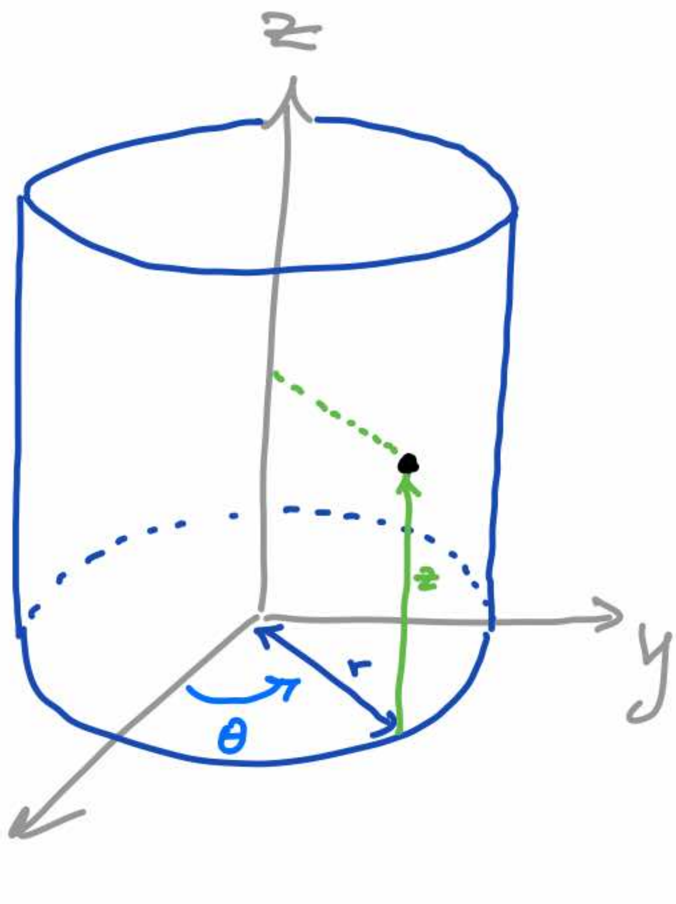
$$= \int_0^2 \left(\int_0^{2-x} (2y - xy - y^2)\, dy \right) dx$$

$$= \int_0^2 \left[y^2 - \tfrac{1}{2} x y^2 - \tfrac{1}{3} y^3 \right]_{y=0}^{2-x} dx$$

$$= \int_0^2 \left(\tfrac{4}{3} - 2x + x^2 - \tfrac{1}{6} x^3 \right) dx$$

$$= \left[\tfrac{4}{3} x - x^2 + \tfrac{1}{3} x^3 - \tfrac{1}{24} x^4 \right]_0^2 = \frac{2}{3}. \quad //$$

Next we make two uses of the change-of-variable formula (Theorem 3).



CYLINDRICAL COORDINATES

We use $\vec{G}(\overset{u_1}{r}, \overset{u_2}{\theta}, \overset{u_3}{z}) =$

$(x(r,\theta,z),\ y(r,\theta,z),\ z(r,\theta,z))$

$:= (r\cos\theta,\ r\sin\theta,\ z).$

So

$$J_{\vec{G}} = \begin{pmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta & \frac{\partial}{\partial z} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta & \frac{\partial}{\partial z} r\sin\theta \\ \frac{\partial}{\partial r} z & \frac{\partial}{\partial \theta} z & \frac{\partial}{\partial z} z \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

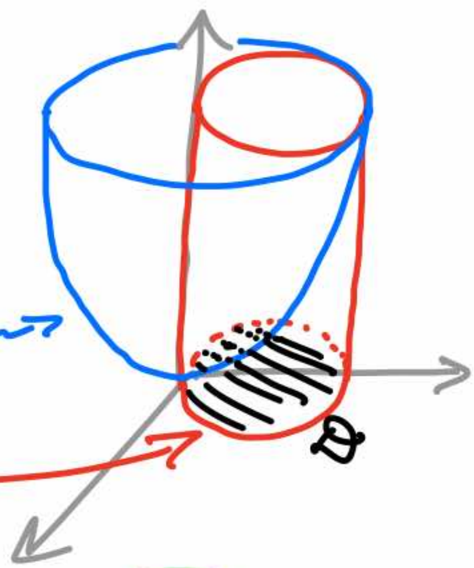
$$\Longrightarrow \left| \det J_{\vec{G}} \right| = r(\cos^2\theta + \sin^2\theta) = r$$

$$\Longrightarrow \int_S f(x,y,z)\, dx\, dy\, dz = \int_T f(\vec{G}(r,\theta,z))\, r\, dr\, d\theta\, dz$$

T i.e. $\vec{G}^{-1}(S)$

Ex 3/ Calculate $\iiint_S z^{-1/2}\, dV$,

where S is the region bounded by $z=0$, $z = x^2+y^2$, & $x^2+y^2 = 2y$.



$$\iiint_S \frac{1}{\sqrt{z}}\, dV = \iint_D \left(\int_0^{x^2+y^2} \frac{1}{\sqrt{z}}\, dz \right) dx\, dy = \int_0^{\pi} \left(\int_0^{2\sin\theta} \left(\int_0^{r^2} \frac{1}{\sqrt{z}}\, dz \right) r\, dr \right) d\theta$$

$x^2+y^2 = 2y$ becomes $r^2 = 2r\sin\theta$ hence $r = 2\sin\theta$

$$= \int_0^{\pi} \left(\int_0^{2\sin\theta} \left[2z^{1/2} \right]_{z=0}^{r^2} r\, dr \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^{2\sin\theta} 2r^2\, dr \right) d\theta = \int_0^{\pi} \frac{16}{3} \sin^3\theta\, d\theta$$

$$= \frac{16}{3} \int_0^{\pi} (\sin\theta - \cos^2\theta \sin\theta)\, d\theta = \frac{16}{3} \left[-\cos\theta + \frac{1}{3}\cos^3\theta \right]_0^{\pi} = \frac{64}{9}.$$

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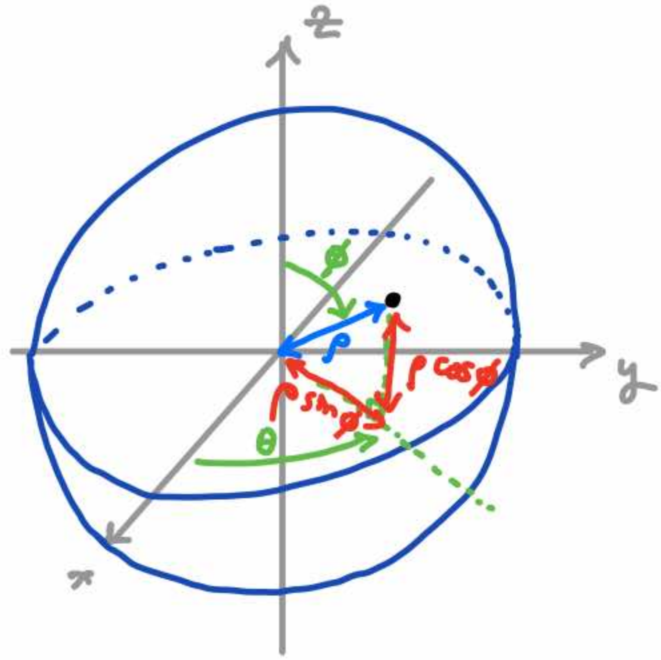
Spherical Coordinates



We use

$\vec{G}(\rho,\theta,\phi) =$

$(\underbrace{\rho\cos\theta\sin\phi}_{x}, \underbrace{\rho\sin\theta\sin\phi}_{y}, \underbrace{\rho\cos\phi}_{z})$

$$\Rightarrow J_{\vec{G}} = \begin{array}{cc} \begin{bmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \\ \begin{matrix} \partial/\partial\rho & \quad \partial/\partial\theta & \quad \partial/\partial\phi \end{matrix} & \end{array}$$

expand along last row

$$\Rightarrow \det J_{\vec{G}} = \cos\phi \cdot \rho\sin\phi \cdot \rho\cos\phi \begin{vmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} \quad -1$$

$$-\rho\sin\phi \cdot \sin\phi \cdot \rho\sin\phi \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \quad +1$$

$$= -\rho^2\sin\phi\left(\cos^2\phi + \sin^2\phi\right)$$

$$= -\rho^2\sin\phi.$$

$$\Rightarrow \left|\det J_{\vec{G}}\right| = \rho^2\sin\phi$$

Meaning: the volume of a little spherical sector is

$$d\rho \cdot \rho\, d\phi \cdot \rho \sin\phi\, d\theta = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi.$$

So $$\int_S f(x,y,z)\, dx\, dy\, dz = \int_{T = \vec{G}^{-1}(S)} f(\vec{G}(\rho,\theta,\phi))\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

Ex $3\frac{1}{2}$/ Volume of a sphere of radius R:

$$\iiint_S 1\, dx\, dy\, dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{R^3}{3} \sin\phi\, d\theta\, d\phi = \int_0^{\pi} \frac{2\pi R^3}{3} \sin\phi\, d\phi$$

$$= \frac{2\pi R^3}{3}\Big[-\cos\phi\Big]_0^{\pi} = \frac{4\pi R^3}{3}. \quad //$$

Ex 4/ Calculate the volume of the n-ball

$$B_n(R) := \left\{ (x_1, \ldots, x_n) \;\middle|\; \sum_{i=1}^{n} x_i^2 \le R^2 \right\}.$$

(this won't use spherical coordinates)

First, $V_n(R) := v(B_n(R)) = \int_{\underbrace{B_n(R)}_{=S}} 1 \, dx_1 \cdots dx_n$

$$= \int_{\underbrace{B_n(1)}_{=T}} 1 \cdot R^n \, du_1 \cdots du_n = R^n V_n(1).$$

linear change of coords:

$\vec{x} = \vec{H}(\vec{u}) = R\vec{u} \quad (x_i = Ru_i)$

$\Rightarrow |J_{\vec{H}}| = R^n$

Next,

$$B_n(1) = \{(x_1, \dots, x_n) \mid x_n^2 + x_{n-1}^2 \leq 1, \; x_1^2 + \cdots + x_{n-2}^2 \leq 1 - x_n^2 - x_{n-1}^2\}$$

$\Downarrow$

$$V_n(1) = \iint_{B_2(1)} \left(\int_{B_{n-2}\left(\sqrt{1-x_n^2-x_{n-1}^2}\right)} 1 \, dx_1 \cdots dx_{n-2} \right) dx_{n-1} \, dx_n$$

$$= \iint_{B_2(1)} V_{n-2}\left(\sqrt{1-x^2-y^2}\right) dx \, dy$$

$$= \iint_{B_2(1)} (1-x^2-y^2)^{\frac{n-2}{2}} V_{n-2}(1) \, dx \, dy$$

$$= V_{n-2}(1) \int_0^{2\pi} \int_0^1 (1-r^2)^{\frac{n}{2}-1} r \, dr \, d\theta$$

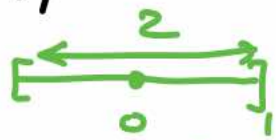
$$= V_{n-2}(1) \cdot 2\pi \cdot \left[-\frac{1}{2} \cdot \frac{2}{n} (1-r^2)^{n/2} \right]_0^1$$

$$= \boxed{\frac{2\pi}{n}} V_{n-2}(1).$$

Finally, set $v_n := \dfrac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$, and notice that

- $\dfrac{v_n}{v_{n-2}} = \dfrac{\pi}{\Gamma(\frac{n}{2}+1)/\Gamma(\frac{n}{2})} = \boxed{\dfrac{2\pi}{n}}$

- $v_1 = \dfrac{\sqrt{\pi}}{\Gamma(3/2)} = \dfrac{\sqrt{\pi}}{\frac{1}{2}\Gamma(\frac{1}{2})} = \dfrac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} = 2 = V_1(1)$



this is $B_1(1)$

- $v_2 = \dfrac{\pi}{\Gamma(2)} = \pi = V_2(1)$

this is $B_2(1)$

Thus $V_n(1) = v_n$ for all n, and

$$V_n(R) = v_n R^n = \frac{\pi^{n/2} R^n}{\Gamma(\frac{n}{2}+1)}.$$

If $n=3$, this is $\dfrac{\pi^{3/2} R^3}{\Gamma(\frac{3}{2}+1)} = \dfrac{\pi\sqrt{\pi}\, R^3}{\frac{3}{4}\sqrt{\pi}} = \dfrac{4}{3}\pi R^3.$