Lecture 43

Surface Area & Surface Integrals
For most of the remainder of this course we shall work in $\mathbb{R}^3$. 
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We know how to integrate on

- curves in $\mathbb{R}^3$
- solid regions in $\mathbb{R}^3$
- what's left is
- surfaces in $\mathbb{R}^3$. 
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Three ways to describe a surface:

- implicitly: as level surface $F(x,y,z)=0$
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Three ways to describe a surface:
- **implicitly**: as level surface $F(x,y,z) = 0$
- **explicitly**: as graph $z = f(x,y)$
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- curves in \( \mathbb{R}^3 \)
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- surfaces in \( \mathbb{R}^3 \).

Three ways to describe a surface:

- **implicitly**: as level surface \( F(x,y,z) = 0 \)
- **explicitly**: as graph \( z = f(x,y) \)
- **parametrically**: \( (u,v) \rightarrow (x(u,v), y(u,v), z(u,v)) \)
  where \( (u,v) \in \mathcal{T} \subseteq \mathbb{R}^2 \).
For most of the remainder of this course we shall work in $\mathbb{R}^3$.

We know how to integrate on
- curves in $\mathbb{R}^3$
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Three ways to describe a surface:
- implicitly: as level surface $F(x,y,z)=0$
- explicitly: as graph $z=f(x,y)$
- parametrically: $(u,v) \mapsto (X(u,v), Y(u,v), Z(u,v))$
  where $(u,v) \in T \subset \mathbb{R}^2$

Ex 1: Cylinder of radius $R$ & height $h$:
\[
\begin{align*}
\text{implicit: } & \quad x^2+y^2 = R^2 \\
\text{explicit: } & \quad y = \pm \sqrt{R^2-x^2} \\
\text{parametric: } & \quad \mathbf{r}(u,v) = (R \cos u, R \sin u, v) \\
\text{where } & \quad (u,v) \in [0,2\pi] \times [0,h]
\end{align*}
\]
For most of the remainder of this course we shall work in $\mathbb{R}^3$.

We know how to integrate on

- curves in $\mathbb{R}^3$;
- solid regions in $\mathbb{R}^3$;
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- implicitly: as level surface $F(x,y,z)=0$
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- parametrically: $(u,v) \rightarrow (X(u,v), Y(u,v), Z(u,v))$ where $(u,v) \in T \subset \mathbb{R}^2$. 

**Ex 1** Cylinder of radius $R$ and height $h$:

- implicit: $x^2+y^2 = R^2$
- explicit: $z = \sqrt{R^2-x^2}$
- parametric: $\hat{r}(u,v) = (R \cos u, R \sin u, v)$ where $(u,v) \in [0,2\pi] \times [0,h]$ 

**Ex 2** Sphere of radius $R$:

- implicit: $x^2+y^2+z^2 = R^2$
- explicit: $z = \sqrt{R^2-x^2-y^2}$
- parametric: $\hat{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)$ where $(u,v) \in [0,2\pi] \times [0,\pi]$

Or $\hat{r}(u,v) = \left( u, v, \sqrt{R^2-u^2-v^2} \right)$, $u^2+v^2 \leq R^2$ for upper hemisphere.
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We know how to integrate on

- curves in $\mathbb{R}^3$;
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- surfaces in $\mathbb{R}^3$.

Three ways to describe a surface:

→ implicitly: as level surface $F(x,y,z) = 0$

→ explicitly: as graph $z = f(x,y)$

→ parametrically: $(u,v) \mapsto (x(u,v), y(u,v), z(u,v))$ where $(u,v) \in T \subset \mathbb{R}^2$.

**Ex 1** Cylinder of radius $R$ and height $h$:

- implicit: $x^2 + y^2 = R^2$
- explicit: $y = \sqrt{R^2-x^2}$
- parametric: $\hat{r}(u,v) = (R\cos u, R\sin u, v)$ where $(u,v) \in [0,2\pi] \times [0,h]$.

**Ex 2** Sphere of radius $R$:

- implicit: $x^2 + y^2 + z^2 = R^2$
- explicit: $z = \sqrt{R^2-x^2-y^2}$
- parametric: $\hat{r}(u,v) = (R\cos u \sin v, R\sin u \sin v, R\cos v)$ where $(u,v) \in [0,2\pi] \times [0,\pi]$

**Ex 3** Cone of vertical angle $\alpha$ and height $h$:

![Diagram of a cone](image)
For most of the remainder of this course we shall work in \( \mathbb{R}^3 \).

We know how to integrate on

- curves in \( \mathbb{R}^3 \)
- solid regions in \( \mathbb{R}^3 \)
- surfaces in \( \mathbb{R}^3 \)

Three ways to describe a surface:

→ implicitly: as level surface \( F(x,y,z)=0 \)

→ explicitly: as graph \( z=f(x,y) \)

→ parametrically: \((u,v) \mapsto (X(u,v), Y(u,v), Z(u,v))\)

where \((u,v) \in \mathcal{T} \subset \mathbb{R}^2 \).

**Ex 1** Cylinder of radius \( R \) & height \( h \):

- implicit: \( x^2+y^2 = R^2 \)
- explicit: \( y = \sqrt{R^2-x^2} \)
- parametric: \( \mathbf{r}(u,v) = (R \cos u, R \sin u, v) \)

where \((u,v) \in [0,2\pi] \times [0,h] \)

**Ex 2** Sphere of radius \( R \):

- implicit: \( x^2+y^2+z^2 = R^2 \)
- explicit: \( z = \sqrt{R^2-x^2-y^2} \)
- parametric: \( \mathbf{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v) \)

where \((u,v) \in [0,2\pi] \times [0,\pi] \)

**Ex 3** Cone of vertical angle \( \alpha \) and height \( h \):

- implicit: \( x^2+y^2 = z^2 \tan^2 \left(\frac{\alpha}{2}\right) \)

\( \frac{z}{h} = \tan \left(\frac{\alpha}{2}\right) \)
Ex 1] Cylinder of radius $R$ & height $h$:
- **Implicit**: $x^2 + y^2 = R^2$
- **Explicit**: $y = \pm \sqrt{R^2 - x^2}$
- **Parametric**: $\vec{r}(u,v) = (R \cos u, R \sin u, v)$ where $(u,v) \in [0, 2\pi] \times [0, h]

Ex 2] Sphere of radius $R$:
- **Implicit**: $x^2 + y^2 + z^2 = R^2$
- **Explicit**: $z = \pm \sqrt{R^2 - x^2 - y^2}$
- **Parametric**: $\vec{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)$ where $(u,v) \in [0, 2\pi] \times [0, \pi]

Ex 3] Cone of vertical angle $\alpha$ and height $h$:
- **Implicit**: $x^2 + y^2 = z^2 \tan^2 \left(\frac{\alpha}{2}\right)$
- **Explicit**: $z = \cot(\alpha) \sqrt{x^2 + y^2}$
- **Parametric**: $\vec{r}(u,v) = (v \sin \alpha \cos u, v \sin \alpha \sin u, v \cos \alpha)$ where $(u,v) \in [0, 2\pi] \times [0, h]$
Let's be more explicit about what sorts of surfaces we want to work with.

Ex 1] Cylinder of radius $R$ & height $h$:

- **implicit**: \( x^2 + y^2 = R^2 \)
- **explicit**: \( y = \pm \sqrt{R^2 - x^2} \) \( \text{and} \ 0 \leq z \leq h \)

**parametric**: \( \vec{r}(u,v) = (R \cos u, R \sin u, v) \)

where \( (u,v) \in [0,2\pi] \times [0,h] \)

Ex 2] Sphere of radius $R$:

- **implicit**: \( x^2 + y^2 + z^2 = R^2 \)
- **explicit**: \( z = \pm \sqrt{R^2 - x^2 - y^2} \)

**parametric**: \( \vec{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v) \)

where \( (u,v) \in [0,2\pi] \times [0,\pi] \)

Ex 3] Cone of vertical angle \( \alpha \) and height \( h \cos \alpha \):

- **implicit**: \( x^2 + y^2 = \tan^2 (\frac{\alpha}{2}) z^2 \)

- **explicit**: \( z = \cot(\alpha) \sqrt{x^2 + y^2} \)

**parametric**: \( \vec{r}(u,v) = (v \sin \alpha \cos u, v \sin \alpha \sin u, v \cos \alpha) \)

where \( (u,v) \in [0,2\pi] \times [0,h] \).
Let's be more explicit about what sorts of surfaces we want to work with. Let:

- \( T \subset \mathbb{R}^2 \) (coordinates: \( u, v \)) be a region with piecewise smooth boundary \( \partial T \).

**Ex 1** Cylinder of radius \( R \) and height \( h \):

- Implicit: \( x^2 + y^2 = R^2 \)
- Explicit: \( y = \pm \sqrt{R^2 - x^2} \)
- Parametric: \( \mathbf{r}(u,v) = (R \cos u, R \sin u, v) \) where \( (u,v) \in [0, 2\pi] \times [0,h] \)

**Ex 2** Sphere of radius \( R \):

- Implicit: \( x^2 + y^2 + z^2 = R^2 \)
- Explicit: \( z = \pm \sqrt{R^2 - x^2 - y^2} \)
- Parametric: \( \mathbf{r}(u,v) = (R \cos u \cos v, R \sin u \cos v, R \sin v) \) where \( (u,v) \in [0, 2\pi] \times [0, \pi] \)

**Ex 3** Cone of vertical angle \( \alpha \) and height \( h \):

- Implicit: \( x^2 + y^2 = z^2 + \tan^2(\frac{\alpha}{2})z \)
- Explicit: \( z = \cot(\alpha) \sqrt{x^2 + y^2} \)
- Parametric: \( \mathbf{r}(u,v) = (v \sin u \cos u, v \sin u \sin u, v \cos u) \) where \( (u,v) \in [0, 2\pi] \times [0, h] \).
Let's be more explicit about what sorts of surfaces we want to work with. Let:

- $T \subset \mathbb{R}^2$ (coordinates: $u, v$) be a region with piecewise smooth boundary $\partial T$;
- $\vec{F} : T \to \mathbb{R}^3$ be a 1-to-1, $C^2$ map;

(technically, we want this to extend to an open set containing $T$)

**Ex 1** Cylinder of radius $R$ & height $h$:

- implicit: $x^2 + y^2 = R^2$
- explicit: $y = \pm \sqrt{R^2 - x^2}$

Parameter: $\vec{F}(u, v) = (R \cos u, R \sin u, v)$ where $(u, v) \in [0, 2\pi] \times [0, h]$

**Ex 2** Sphere of radius $R$:

- implicit: $x^2 + y^2 + z^2 = R^2$
- explicit: $z = \pm \sqrt{R^2 - x^2 - y^2}$

Parameter: $\vec{F}(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)$ where $(u, v) \in [0, 2\pi] \times [0, \pi]$

**Ex 3** Cone of vertical angle $\alpha$ and height $h$:

- implicit: $x^2 + y^2 = z^2 \tan^2 \alpha$
- explicit: $z = \frac{x}{\sin \alpha} \sqrt{x^2 + y^2}$

Parameter: $\vec{F}(u, v) = (v \sin \alpha \cos u, v \sin \alpha \sin u, v \alpha)$

$(u, v) \in [0, 2\pi] \times [0, h]$.
Let's be more explicit about what sorts of surfaces we want to work with. Let:

- $T \subseteq \mathbb{R}^2$ (coordinates: $u, v$) be a region with piecewise smooth boundary $\partial T$;
- $\tilde{F} : T \to \mathbb{R}^3$ be a 1-to-1, $C^1$ map;
- $\tilde{S} := \tilde{F}(T)$ the image of $\tilde{F}$.

**Ex 1** Cylinder of radius $R$ and height $h$:
- implicit: $x^2 + y^2 = R^2$
- explicit: $y = \sqrt{R^2 - x^2}$ (and $0 \leq z \leq h$)
- parameter: $\tilde{F}(u,v) = (R \cos u, R \sin u, v)$ when $(u,v) \in [0, 2\pi] \times [0, h]$.

**Ex 2** Sphere of radius $R$:
- implicit: $x^2 + y^2 + z^2 = R^2$
- explicit: $z = \sqrt{R^2 - x^2 - y^2}$
- parameter: $\tilde{F}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)$ where $(u,v) \in [0, 2\pi] \times [0, \pi]$.

**Ex 3** Cone of vertical angle $\alpha$ and height $h$:
- implicit: $x^2 + y^2 = z^2 \tan^2 \left(\frac{\pi}{2} - \frac{\alpha}{2}\right)$
- explicit: $z = \cot(\alpha) \sqrt{x^2 + y^2}$
- parameter: $\tilde{F}(u,v) = (v \sin \alpha \cos u, v \sin \alpha \sin u, v \cos \alpha)$ where $(u,v) \in [0, 2\pi] \times [0, h]$.
Let's be more explicit about what sorts of surfaces we want to work with. Let:

1. \( T \subseteq \mathbb{R}^2 \) (coordinates: \( u,v \)) be a region with piecewise smooth boundary \( \partial T \);
2. \( \hat{\hat{r}} : T \rightarrow \mathbb{R}^3 \) be a 1-to-1, \( C^2 \) map;
3. \( \hat{r} := \hat{\hat{r}}(T) \) the image of \( \hat{\hat{r}} \).

**Definition:** \( \hat{r} \) is a **smooth parameterization** of \( \hat{\hat{r}} \) if \( \hat{r}_u \times \hat{r}_v \) is never zero.

\( \hat{\hat{r}} \) is a **smooth surface** if it has a smooth parametrization (or a union of "open subsets" with such parametrizations).

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**Ex 1** Cylinder of radius \( R \) & height \( h \):

- **Implicit**:
  
  \[
  x^2 + y^2 = R^2 \quad \text{(and } 0 \leq z \leq h)\]

- **Explicit**:
  
  \[
  y = \sqrt{R^2 - x^2} \]

- **Parametric**:
  
  \[
  \hat{r}(u,v) = (R \cos u, R \sin u, v) \quad \text{where } (u,v) \in [0,2\pi] \times [0,h] \]

---

**Ex 2** Sphere of radius \( R \):

- **Implicit**:
  
  \[
  x^2 + y^2 + z^2 = R^2 \]

- **Explicit**:
  
  \[
  z = \pm \sqrt{R^2 - x^2 - y^2} \]

- **Parametric**:
  
  \[
  \hat{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v) \quad \text{where } (u,v) \in [0,2\pi] \times [0,\pi] \]

---

**Ex 3** Cone of vertical angle \( \alpha \) and height \( h \):

- **Implicit**:
  
  \[
  x^2 + y^2 = z^2 \tan^2 \frac{\alpha}{2} \quad \text{(at } h \text{)} \]

- **Explicit**:
  
  \[
  z = \cot(\alpha) \sqrt{x^2+y^2} \]

- **Parametric**:
  
  \[
  \hat{r}(u,v) = (v \sin u \cos \alpha, v \sin u \sin \alpha, v \cos \alpha) \quad \text{where } (u,v) \in [0,2\pi] \times [0,h] \]
Let's be more explicit about what sorts of surfaces we want to work with. Let:

- \( T \subset \mathbb{R}^2 \) (coordinates: \( u, v \)) be a region with piecewise smooth boundary \( \partial T \);  
- \( \hat{\varphi} : T \to \mathbb{R}^3 \) be a 1-to-1, \( C^1 \) map; (technically, we want this to extend to an open set containing \( T \)); and
- \( \hat{\mathcal{S}} := \hat{\varphi}(T) \) the image of \( \hat{\varphi} \).

**Definition:** \( \hat{\varphi} \) is a smooth parametrization of \( \hat{\mathcal{S}} \) if \( \hat{\varphi}_u \times \hat{\varphi}_v \) is never zero.\( \hat{\mathcal{S}} \) is a smooth surface if it has a smooth parametrization (or a union of "open subsets" with such parametrizations).

**Example:** One can show that \( \mathcal{S} = \{ (y, y) \mid F(x, y, z) = 0 \} \) is smooth if at every point on it \( \hat{\varphi} \) has a non-vanishing partial ("Implicit Function Thm.").
Let's be more explicit about what sorts of surfaces we want to work with. Let:

- \( T \subset \mathbb{R}^2 \) (coordinates: \( u, v \)) be a region with piecewise smooth boundary \( \partial T \);
- \( \tilde{F} : T \to \mathbb{R}^3 \) be a 1-to-1, \( C^2 \) map;
- \( \hat{T} := \tilde{F}(T) \) the image of \( \tilde{F} \).

**Definition:** \( \hat{T} \) is a smooth parametrization of \( \hat{S} \) if \( \tilde{F}_u \times \tilde{F}_v \) is never zero.

\( \hat{S} \) is a smooth surface if it has a smooth parametrization (or a union of "open subsets" with such parametrizations).

**Notes:** Write \( \tilde{F}(u,v) = (X(u,v), Y(u,v), Z(u,v)) \).

1. The condition in the Definition is the same as asking

\[
J_{\tilde{F}} = \begin{pmatrix}
X_u & X_v \\
Y_u & Y_v \\
Z_u & Z_v
\end{pmatrix}
\]

to have rank 2 everywhere. (Why?)

**Example:**

One can show that \( \hat{S} = \{(x,y) \mid F(x,y,z) = 0\} \) is smooth if at every point on \( \hat{S} \) \( F \) has a non-vanishing partial ("Implicit Function Thm.").
Let's be more explicit about what sorts of surfaces we want to work with. Let:

- \( T \subset \mathbb{R}^2 \) (coordinates: \( u, v \)) be a region with piecewise smooth boundary \( \partial T \);
- \( \vec{T}: T \to \mathbb{R}^3 \) be a 1-to-1, \( C^2 \) map; (technically, we want this to extend to an open set containing \( T \)) and
- \( \bar{\Delta} := \vec{T}(T) \) the image of \( \vec{T} \).

**Definition:** \( \bar{\Delta} \) is a smooth parametrization of \( \Delta \) if \( \vec{\partial}_u \times \vec{\partial}_v \) is never zero.

\( \Delta \) is a smooth surface if it has a smooth parametrization (or a union of "open subsets" with such parametrizations).

**e.g.** One can show that \( \Delta = \{ (x,y,z) \mid F(x,y,z) = 0 \} \) is smooth if at every point on it \( F \) has a non-vanishing partial ("Implicit Function Thm.").

**Notes:** Write \( \vec{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)) \).

1. The condition in the Definition is the same as asking
   \[ J_{\vec{T}} = \begin{vmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{vmatrix} \]
   to have rank 2 everywhere. (Why?)

2. \( \vec{\partial}_u \times \vec{\partial}_v \) should be viewed as a vector normal to \( \Delta \), with length equal to the ratio of areas \( a(\vec{T}(\bar{\Delta})) / a(\bar{\Delta}) \).
Notes: Write $\mathbf{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$.

1. The condition in the Definition is the same as asking
   \[
   J_\mathbf{r} = \begin{pmatrix}
   X_u & X_v \\
   Y_u & Y_v \\
   Z_u & Z_v 
   \end{pmatrix}
   \]
   to have rank 2 everywhere. (Why?)

2. $\mathbf{r}_u \times \mathbf{r}_v$ should be viewed as a vector normal to $S$, with length equal to the ratio of areas $a(\mathbf{r}(\mathbf{R}_j))/a(\mathbf{R}_j)$.

3. If $S$ is described “explicitly” by $z = f(x,y)$, $(x,y) \in \mathbf{D} \subset xy$-plane, then it is described parametrically by $\mathbf{r}(x,y) = (x,y,f(x,y))$. ($\mathbf{D} : (u,v) = (x,y), \mathbf{R} = S$.)
Ex 4 \[ \text{For the sphere, with} \]
\[ \hat{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v), \]
we have
\[ \hat{r}_u = (-R \sin u \sin v, R \cos u \sin v, 0) \]
\[ \hat{r}_v = (R \cos u \cos v, R \sin u \cos v, -R \sin v) \]

Notes: Write \[ \hat{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)). \]

1. The condition in the Definition is the same as asking
\[ J_r = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix} \]
to have rank 2 everywhere. (Why?)

2. \[ \hat{r}_u \times \hat{r}_v \] should be viewed as a vector normal to \( S \), with length equal to the ratio of areas \( a(\hat{r}(R_{ij})) / a(R_{ij}) \) :

3. If \( S \) is described “explicitly” by \( z = f(x,y) \), \((x,y) \in \Omega \subset xy\)-plane, then it is described parametrically by \( \hat{r}(x,y) = (x,y,f(x,y)), (x_0 : (u,v) = (x,y), \theta = \)
Ex 4: For the sphere, with
\[ \mathbf{r}(u,v) = (R \cos u \cos v, R \sin u \cos v, R \sin v), \]
we have
\[ \mathbf{r}_u = (-R \sin u \sin v, R \cos u \cos v, 0), \]
\[ \mathbf{r}_v = (R \cos u \cos v, R \sin u \cos v, -R \sin v). \]

\[ \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = (-R^2 \cos u \sin^2 v, -R^2 \sin u \sin^2 v, -R^2 \sin v \cos v) \]
\[ = -R \sin v \mathbf{r}(u,v). \]

Notes: Write \( \mathbf{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)) \).

1. The condition in the Definition is the same as asking
\[ J_F^2 = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix} \]
to have rank 2 everywhere. (Why?)

2. \( \mathbf{r}_u \times \mathbf{r}_v \) should be viewed as a vector normal to \( \mathcal{S} \), with length equal to the ratio of areas \( a(\mathbf{r}(R_i))/a(R_i) \):

3. If \( \mathcal{S} \) is described "explicitly" by \( z = f(x,y) \), \((x,y) \in \mathcal{D} \subseteq xy\)-plane, then it is described parametrically by \( \mathbf{r}(x,y) = (x, y, f(x,y)) \), \((x,y) \in \mathcal{D} \).
Ex 4 | For the sphere, with
\[ \vec{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v) , \]
we have
\[ \vec{r}_u = (-R \sin u \sin v, R \cos u \sin v, 0) \]
\[ \vec{r}_v = (R \cos u \cos v, R \sin u \cos v, -R \sin v) \]
\[ \Rightarrow \vec{r}_u \times \vec{r}_v = (-R^2 \cos u \sin^2 v, -R^2 \sin u \sin^2 v, -R^2 \sin v \cos v) \]
\[ = -R \sin v \vec{r}(u,v) \]
\[ \Rightarrow ||\vec{r}_u \times \vec{r}_v|| = R^2 \sin v. \quad \text{(on sphere !)} \]

Notes: Write \( \vec{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)) \).

1. The condition in the Definition is the same as asking
\[ J^2_f = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix} \]
to have rank 2 everywhere. (Why?)

2. \( \vec{r}_u \times \vec{r}_v \) should be viewed as
a vector normal to \( \Delta \), with length equal to
the ratio of areas \( \Delta_f / \Delta_g \) :

3. If \( \Delta \) is described "explicitly"
by \( z = f(x,y) ; (x,y) \in \Delta \subset xy\)-plane,
then it is described parametrically by
\[ \vec{r}(x,y) = (x,y,f(x,y)) \quad (x,y) \in \Delta \]
so \( (u,v) = (x,y), \quad \Delta = \Delta_e \).
Ex 4. For the sphere, with
\[ \hat{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v), \]
we have
\[ \hat{r}_u = (-R \sin u \sin v, R \cos u \sin v, 0), \]
\[ \hat{r}_v = (R \cos u \cos v, R \sin u \cos v, -R \sin v). \]
\[ \Rightarrow \hat{r}_u \times \hat{r}_v = (-R^2 \cos u \sin^2 v, -R^2 \sin u \sin^2 v, -R^2 \sin v \cos v) \]
\[ = -R \sin v \hat{r}(u,v). \]
\[ \Rightarrow ||\hat{r}(u,v)|| = R. \]
(On sphere!) \[ \Rightarrow ||\hat{r}_u \times \hat{r}_v|| = R^2 \sin v. \]

Note that this is singular at \( v = 0 \) and \( \pi \) (the north & south poles), and not 1-1 at \( u = 0 \) \& \( 2\pi \). But it is this happens on a set of "content zero" in \( \mathcal{T} = [0,2\pi] \times [0,\pi], \) it isn't a problem for using the parameterization \( \hat{r} \) to integrate over \( \Sigma. \)

Notes:
- Write \( \hat{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)). \)

1. The condition in the Definition is the same as asking
\[ J_\hat{r} = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix} \]
to have rank 2 everywhere. (Why?)

2. \( \hat{r}_u \times \hat{r}_v \) should be viewed as a vector normal to \( \Sigma, \) with length equal to the ratio of areas \( a(\hat{r}(R_j)) / a(R_j). \)

3. If \( \Sigma \) is described "explicitly" by \( z = f(x,y), (x,y) \in \mathcal{S} \subset xy\)-plane, then it is described parametrically by
\[ \hat{r}(x,y) = (x,y,f(x,y)), \text{ (so: } (u,v) = (x,y), \mathcal{T} = S) \]
A bit more on ② & ③:

Notes: Write \( \mathbf{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)) \).

1. The condition in the Definition is the same as asking

   \[
   J_r^2 = \begin{pmatrix}
   X_u & Y_u \\
   X_v & Y_v \\
   Z_u & Z_v
   \end{pmatrix}
   \]

   to have rank 2 everywhere. (Why?)

2. \( \mathbf{r}_u \times \mathbf{r}_v \) should be viewed as a vector normal to \( S \), with length equal to the ratio of areas \( a(\mathbf{r}(R_{ij})) / a(R_{ij}) \):

3. If \( S \) is described “explicitly” by \( z = f(x,y) \), \( (x,y) \in \Omega \subset xy\)-plane, then it is described parametrically by

   \[
   \mathbf{r}(x,y) = (x,y, f(x,y)). \quad \text{So:} \quad (u,v) = (x,y), \quad \Omega = \mathbb{D}.
   \]
A bit more on $\mathbf{3}$:

- One way to think about $\mathbf{1}^i(\mathbf{u}, \mathbf{v}) \times \mathbf{2}^i(\mathbf{u}, \mathbf{v})$ being $1$ to $\mathcal{S}$ at $\mathbf{r}(\mathbf{u}, \mathbf{v})$ is by showing it is $1$ to the tangent vector of any curve $C \subset \mathcal{S}$ passing through this point.

Notes:

Write $\mathbf{r}(\mathbf{u}, \mathbf{v}) = (X(\mathbf{u}, \mathbf{v}), Y(\mathbf{u}, \mathbf{v}), Z(\mathbf{u}, \mathbf{v}))$.

1. The condition in the definition is the same as asking

   \[ J_\mathbf{r} = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix} \]

   to have rank $2$ everywhere. (Why?)

2. $\mathbf{1}^i \times \mathbf{2}^i$ should be viewed as a vector normal to $\mathcal{S}$, with length equal to the ratio of areas $a(\mathbf{r}(\mathbf{R}_j))/a(\mathbf{R}_j)$:

3. If $\mathcal{S}$ is described "explicitly" by $z = f(x, y)$, $(x, y) \in \mathcal{D} \subset \mathbf{x}_y$-plane, then it is described parametrically by

   $\mathbf{r}(x, y) = (x, y, f(x, y))$, $(x, y) \in \mathcal{D} = \mathcal{D}.$
A bit more on \( \mathbf{2} \) and \( \mathbf{3} \):

- One way to think about \( \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0) \) being \( \perp \) to \( S \) at \( \vec{r}(u_0,v_0) \) is by showing it is \( \perp \) to the tangent vector of any curve \( C \subset S \) passing through this point; view \( C = \vec{r}(C^*) \) for \( C^* \subset T \) parametrized by \( \tilde{t}(t) \), so that \( \tilde{t} \) parametrizes \( C \).

**Notes:** Write \( \vec{r}(u,v) = (X(u,v), Y(u,v), Z(u,v)) \).

1. The condition in the Definition is the same as asking
   \[
   J_{\vec{r}} = \begin{pmatrix}
   X_u & X_v \\
   Y_u & Y_v \\
   Z_u & Z_v
   \end{pmatrix}
   \]
   to have rank 2 everywhere. (Why?)

2. \( \vec{r}_u \times \vec{r}_v \) should be viewed as a vector normal to \( S \), with length equal to the ratio of areas \( a(\vec{r}(R_{ij})) / a(R_{ij}) \):

3. If \( S \) is described “explicitly” by \( z = f(x,y) \), \((x,y) \in D \subset xy\)-plane, then it is described parametrically by \( \vec{r}(x,y) = (x,y,f(x,y)) \). (So: \((u,v) = (x,y), \partial = D\).)
A bit more on $\partial_1 \partial_3$:

1. One way to think about $\partial_u(u_0, v_0) \times \partial_v(u_0, v_0)$ being $\perp$ to $\mathcal{S}$ at $\vec{r}(u_0, v_0)$ is by showing it is $\perp$ to the tangent vector of any curve $C \subset \mathcal{S}$ passing through this point: View $C = \vec{r}(C^t)$ for $C^t \in \mathcal{T}$ parametrized by $\vec{a}(t)$, so that $\partial_\vec{a}^2$ parametrizes $C \subset \mathcal{S}$, then

$$\vec{a}'(t) = \left( \frac{u'(t)}{v'(t)} \right) \text{ and } (\vec{a}^2)'(t) = \left( \frac{u''(t)}{v''(t)} \right) = u'(t) \vec{e}_v + v'(t) \vec{e}_u$$

is $\perp$ to $\partial_u \times \partial_v$.

Notes:

1. The condition in the Definition is the same as asking

$$J_\vec{r} = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix}$$

to have rank 2 everywhere. (Why?)

2. $\vec{r}_u \times \vec{r}_v$ should be viewed as a vector normal to $\mathcal{S}$, with length equal to the ratio of areas $\alpha(\vec{r}(R_{ij}))/\alpha(R_{ij})$.

3. If $\mathcal{S}$ is described "explicitly" by $z = f(x, y), (x, y) \in \mathcal{D} \subset xy$-plane, then it is described parametrically by $\vec{r}(x, y) = (x, y, f(x, y)), (x, : (u, v) = (x, y), \mathcal{T} = \mathcal{D})$. 
A bit more on (2) and (3):

- one way to think about $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$ being $\perp$ to $\Delta$ at $\vec{r}(u_0,v_0)$ is by showing it is $\perp$ to the tangent vector of any curve $C \subset \Delta$ passing through this point: view $C = \vec{r}(C^t)$ for $C^t \subset T$ parametrized by $\vec{d}(t)$, so that $\vec{d}(0)$ parameterizes $C$:  
  $$\vec{d}(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \quad \text{and} \quad (\vec{d}(t))'(t) = \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} \left( \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix} \right) = u'(t) \vec{r}_u + v'(t) \vec{r}_v$$

  is $\perp$ to $\vec{r}_u \times \vec{r}_v$.

- In (3), $\vec{r}_u = \vec{r}_x = (1, 0, f_x)$, $\vec{r}_v = \vec{r}_y = (0, 1, f_y)$, $\Rightarrow \vec{r}_u \times \vec{r}_v = (-f_x, -f_y, 1)$.

Notes:

Write $\vec{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$.

1. The condition in the Definition is the same as asking

   $$J_r = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix}$$

   to have rank 2 everywhere. (Why?)

2. $\vec{r}_u \times \vec{r}_v$ should be viewed as a vector normal to $\Delta$, with length equal to the ratio of areas $a(\vec{r}(R_j))/a(R_{ij})$.

3. If $\Delta$ is described "explicitly" by $z = f(x,y), (x,y) \in \Omega \subset xy$-plane, then it is described parametrically by $\vec{r}(x,y) = (x,y,f(x,y)), \text{so: } (u,v) = (x,y), \Omega = \Delta.$
A bit more on $\mathbb{R}^3$:

- One way to think about $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$ being $\perp$ to $\mathcal{S}$ at $\vec{r}(u_0,v_0)$ is by showing it is $\perp$ to the tangent vector of any curve $C \subseteq \mathcal{S}$ passing through this point: view $C = \vec{r}(\mathbb{R}^2)$ for $\mathbb{R}^2 \subseteq T$ parametrized by $\vec{r}(t)$, so that $\vec{r} \circ \vec{a}$ parametrizes $C$: then
  \[
  \vec{a}'(t) = \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} \quad \text{and} \quad (\vec{r} \circ \vec{a})'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = (\vec{r}_u \circ \vec{a})(v'(t)) = u'(t) \vec{r}_u + v'(t) \vec{r}_v
  \]
  is $\perp$ to $\vec{r}_u \times \vec{r}_v$.

- In $\mathbb{R}^3$, $\vec{r}_u = \vec{r}_x = (1, 0, f_x)$, $\vec{r}_v = \vec{r}_y = (0, 1, f_y)$
  \[\Rightarrow \vec{r}_u \times \vec{r}_v = (-f_x, -f_y, 1).\]

- If $f(x,y)$ is defined implicitly by
  \[F(x,y, f(x,y)) = 0,\]
  then recall applying $\frac{\partial}{\partial x}$
  \[F_x + f_x F_z = 0 \Rightarrow f_x = -F_z/F_x \text{ etc.} \]

**Notes:** Write $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.

1. The condition in the Definition is the same as asking
   \[
   J_r = \begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix}
   \]
   to have rank 2 everywhere. (Why?)

2. $\vec{r}_u \times \vec{r}_v$ should be viewed as a vector normal to $\mathcal{S}$, with length equal to the ratio of areas $\alpha(\vec{r}(\mathbb{R}^2))/\alpha(\mathbb{R}^2)$:

3. If $\mathcal{S}$ is described "explicitly" by $z = f(x,y), (x,y) \in \mathcal{D} \subseteq xy$-plane, then it is described parametrically by $\vec{r}(x,y) = (x,y,f(x,y)), (x_0 : (u,v) = (x,y), \mathcal{D} = \mathbb{R}^2)$.
A bit more on ② & ③:

- One way to think about $\vec{r}_u'(u_0,v_0) \times \vec{r}_v'(u_0,v_0)$ being $\perp$ to $\vec{g}$ at $\vec{r}(u_0,v_0)$ is by showing it is $\perp$ to the tangent vector of any curve $C \in \Delta$ passing through this point: view $C = \vec{r}(c^t)$ for $c^t \in T$ parameterized by $\vec{r}(c^t)$, so that $\vec{r} \circ c$ parameterizes $C$. Then
  $$\vec{r}'(c^t) = \left( \frac{\partial}{\partial c^t} \right) \left( \vec{r}(c^t) \right) = \left( \frac{\partial}{\partial c^t} \vec{r}_u \right) \left( \vec{r}(c^t) \right) + \left( \frac{\partial}{\partial c^t} \vec{r}_v \right) \left( \vec{r}(c^t) \right)$$
  $$= u(c^t) \vec{r}_u + v(c^t) \vec{r}_v$$
  is $\perp$ to $\vec{r}_u \times \vec{r}_v$.

- In ③, $\vec{r}_u = \frac{\partial}{\partial u} = (1, 0, f_x)$, $\vec{r}_v = \frac{\partial}{\partial v} = (0, 1, f_y)$
  $\Rightarrow \vec{r}_u \times \vec{r}_v = (-f_x, -f_y, 1)$.

- If $f(x,y)$ is defined "implicitly" by
  $$F(x,y,f(x,y)) = 0$$
  then recall applying $\frac{\partial}{\partial x} \Rightarrow$  
  $$F_x + f_x F_y = 0 \Rightarrow f_x = -\frac{F_y}{F_x} \text{ etc.}$$
  $\Rightarrow \vec{r}_u \times \vec{r}_v = (\frac{F_y}{F_x}, \frac{F_x}{F_y}, 1) = \frac{1}{F_x} \vec{V} F.$

Notes: Write $\vec{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$.

1. The condition in the Definition is the same as asking
   $$J_\vec{r} = \begin{pmatrix}
   X_u & X_v \\
   Y_u & Y_v \\
   Z_u & Z_v
   \end{pmatrix}$$
   to have rank 2 everywhere. (Why?)

2. $\vec{r}_u \times \vec{r}_v$ should be viewed as a vector normal to $\Delta$, with length equal to the ratio of areas $a (\vec{r}(\alpha)) / a(\vec{r}(\beta))$.

3. If $\Delta$ is described "explicitly" by $z = f(x,y)$, $(x,y) \in \Omega \subset xy$-plane, then it is described parametrically by
   $$\vec{r}(x,y) = (x, y, f(x,y)),$$
   $(\Omega : (u,v) = (x,y), \alpha = \beta)$. 

}\!
Summary

- Parameter: \((x, y, z) = \vec{r}(u, v) \Rightarrow \vec{v}_u \times \vec{v}_v \) normal to \(S\)
- \(\vec{r}(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\)
  \[ \vec{v}_u \times \vec{v}_v = -R \sin v \vec{v}(u, v) \text{ for sphere} \]
- Explicit: \(z = f(x, y) \Rightarrow \vec{r}(x, y) = (x, y, f(x, y))\)
  \[ \vec{v}_x \times \vec{v}_y = (-f_x, -f_y, 1) \]
- Implicit: \(F(x, y, z) = 0 \Rightarrow \vec{v}_x \times \vec{v}_y = \frac{1}{F_z} (F_x, F_y, F_z)\).
**Summary**

- Parameterization: \((x, y, z) = \mathbf{r}(u, v) \Rightarrow \mathbf{r}_u \times \mathbf{r}_v\) normal to \(S\)

  e.g. \([\mathbf{r}(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\]

  \(\mathbf{r}_u \times \mathbf{r}_v = -R \sin v \mathbf{r}(u, v)\) for sphere

- Explicit: \(z = F(x, y) \Rightarrow \mathbf{r}(x, y) = (x, y, F(x, y))\)

  \(\mathbf{r}_x \times \mathbf{r}_y = (-F_y, -F_x, 1)\)

- Implicit: \(F(x, y, z) = 0 \Rightarrow \mathbf{r}_x \times \mathbf{r}_y = \frac{1}{F_z}(F_x, F_y, F_z)\).

Since \(\|\mathbf{r}_x \times \mathbf{r}_y\| dudv\) was the area of the parallelogram approximating \(\mathbf{F}(R_{ij})\), it makes sense to define the surface area of \(S\) by

\[
\mathcal{A}(S) := \iint_S \|\mathbf{r}_x \times \mathbf{r}_y\| dudv
\]

in the parametric setting.
**Summary**

- Parameter: \((x, y, z) = \hat{r}(u, v) \Rightarrow \hat{r}_u \times \hat{r}_v\) normal to \(S\)

  e.g. \(\hat{r}(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\)
  \[\hat{r}_u \times \hat{r}_v = -R \sin v \hat{r}(u, v)\] for sphere

- Explicit: \(z = F(x, y) \Rightarrow \hat{r}(x, y) = (x, y, F(x, y))\)
  \[\hat{r}_u \times \hat{r}_v = (-f_x, -f_y, 1)\]

- Implicit: \(F(x, y, z) = 0 \Rightarrow \hat{r}_u \times \hat{r}_v = \frac{1}{\sqrt{F_x^2 + F_y^2 + F_z^2}} (F_x, F_y, F_z)\).

Since \(\|\hat{r}_u \times \hat{r}_v\| \mathrm{d}u \mathrm{d}v\) was the area of the parallelogram approximating \(F(R)\), it makes sense to define the **surface area of \(S\)** by

\[\alpha(\mathcal{S}) := \iint_S \|\hat{r}_u \times \hat{r}_v\| \mathrm{d}u \mathrm{d}v\]

in the parametric setting.

- Explicit setting: \[\iint_S \sqrt{1 + f_x^2 + f_y^2} \, \mathrm{d}u \mathrm{d}y\]
**Summary**

- Parameter: \((x,y,z) = \hat{r}(u,v)\) \(\Rightarrow\) \(\hat{F}_u \times \hat{F}_v\) normal to \(S\)
  - e.g. \(\hat{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\)
  - \(\hat{r}_u \times \hat{r}_v = -R \sin v \hat{r}(u,v)\) for sphere
- Explicit: \(z = F(x,y)\) \(\Rightarrow\) \(\hat{r}(x,y) = (x,y,F(x,y))\)
  - \(\hat{F}_x \times \hat{F}_y = (-F_y, -F_x, 1)\)
- Implicit: \(F(x,y,z) = 0\) \(\Rightarrow\) \(\hat{F}_x \times \hat{F}_y = \frac{1}{F_z} (F_x, F_y, F_z)\).

Since \(\|\hat{F}_u \times \hat{F}_v\|\) was the area of the parallelogram approximating \(F(R_{ij})\), it makes sense to define the surface area of \(S\) by:

\[
\alpha(S) := \iiint_S \|\hat{F}_u \times \hat{F}_v\| \, du \, dv
\]

in the parametric setting.

- Explicit setting: \(\iint_S \sqrt{1 + F_x^2 + F_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)
**Summary**

- Parameter: \((x, y, z) = \hat{r}(u, v) = F_u \times F_v\) normal to \(S\)
  - Explicit: \(F(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\)
  - Implicit: \(F_x \times F_y = -R \sin v \hat{r}(u, v)\) for sphere

- Explicit: \( \hat{r} = F(x, y) \Rightarrow \hat{r}(x, y) = (x, y, f(x, y))\)
  - \(F_x \times F_y = (-F_y, -F_x, 1)\)
- Implicit: \(F(x, y, z) = 0 \Rightarrow F_x \times F_y = \frac{1}{F_z} (F_x, F_y, F_z)\)

---

Since \(\|F_u \times F_v\|\) was the area of the parallelogram approximating \(F(R, \theta)\), it makes sense to define the surface area of \(S\) by

\[
a(S) := \iint_S \|F_u \times F_v\| \, du \, dv
\]

in the parameter setting.

- Explicit setting: \(\iint_S \sqrt{1 + F_x^2 + F_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)

---

**Ex 5** We find the surface area of the upper hemisphere of radius \(R\) in two ways:
**Summary**

- Parameterized: \((x,y,z) = \mathbf{r}(u,v) = F_x \mathbf{u} + F_y \mathbf{v} \) normal to \(S\)
- \( \mathbf{r}(u,v) = (R \cos u \sin v, R \sin u \sin v, R \cos v) \)
- \( \mathbf{F} \times \mathbf{v} = -R \sin v \mathbf{r}(u,v) \) for sphere
- Explicit: \( z = F(x,y) \Rightarrow \mathbf{r}(x,y) = (x,y,F(x,y)) \)
  \( \mathbf{F} \times \mathbf{y} = (-F_x, -F_y, 1) \)
- Implicit: \( F(x,y,z) = 0 \Rightarrow \mathbf{F} \times \mathbf{y} = \frac{1}{F_z} (F_x, F_y, F_z) \).

Since \( \| \mathbf{F} \times \mathbf{v} \| \) was the area of the parallelogram approximating \( F(R; \mathbf{y}) \), it makes sense to define the **surface area** of \( S \) by

\[
\alpha(S) = \iint_{S} \| F_x \times \mathbf{v} \| \, dudv
\]

in the parameter setting.

- **Explicit setting**: \( \iint_{S} \sqrt{1 + F_x^2 + F_y^2} \, dxdy \)
- **Implicit setting**: \( \iint_{S} \sqrt{F_x^2 + F_y^2 + F_z^2} \, dxdy \)

---

**Ex 5**  We find the surface area of the upper hemisphere of radius \( R \) in two ways:

- Parameterically:
  \[
  \iint_{S} R \sin v \sqrt{\| \mathbf{r}(u,v) \|^2} \, dudv
  \]
  \( S = [0,2\pi] \times [0,\pi/2] \)
  \[
  = R^2 \int_{0}^{2\pi} \int_{0}^{\pi/2} \sin v \, dv \, du
  \]
  \[
  = 2\pi R^2.
  \]

- Explicitly:
  \[
  \iint_{S} \sqrt{1 + F_x^2 + F_y^2} \, dxdy
  \]
  \( S \) is the hemisphere.
**Summary**
- Parameterization: \((x, y, z) = \vec{r}(u, v) = F_u \times F_v\) normal to \(S\)
  - \(F_u \times F_v = (-R \sin u \cos v, R \sin u \sin v, R \cos v)\) for sphere
- Explicit setting: \(z = F(x, y) \Rightarrow \vec{r}(x, y) = (x, y, f(x, y))\)
  - \(F_u \times F_y = (-F_x, -F_y, 1)\)
- Implicit setting: \(F(x, y, z) = 0 \Rightarrow \vec{r}_x \times \vec{r}_y = \frac{1}{F_z}(F_x, F_y, F_z)\).

Since \(\|F_u \times F_v\|\) was the area of the parallelogram approximating \(\vec{F}(R, i, j)\), it makes sense to define the surface area of \(S\) by:

\[
\sigma(S) = \iint_S \|F_u \times F_v\| \, du \, dv
\]

in the parametric setting.

- Explicit setting: \(\iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_D \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)

**Ex 5**
We find the surface area of the upper hemisphere of radius \(R\) in two ways:

1. Parametrically:
   \[
   \iint_S R \sin v \|\vec{r}(u, v)\| \, du \, dv
   \]
   \(S = \{v \in [0, 2\pi] \times [0, \frac{\pi}{2}]\}\)
   \[
   = R^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \sin v \, dv \, du
   \]
   \[
   = 2\pi R^2.
   \]

2. Implicitly:
   \[D = \text{disk of radius } R, \quad F(x, y, z) = x^2 + y^2 + z^2 - R^2\]
   \[
   \iint_D \sqrt{(2x)^2 + (2y)^2 + (2z)^2} \, dx \, dy
   \]
   \[
   = \iint_D \frac{2R}{z} \, dx \, dy
   \]
   \[
   = \sqrt{R^2 - z^2}.
   \]
**Summary**
- Parameter: \((x, y, z) = \vec{r}(u, v) \Rightarrow \vec{r}_u \times \vec{r}_v\) normal to \(S\)
  - \(\vec{r}(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\)
  - \(\vec{r}_u \times \vec{r}_v = -R \sin v \vec{r}(u, v)\) for sphere
- Explicit: \(z = f(x, y) \Rightarrow \vec{r}(x, y) = (x, y, f(x, y))\)
  - \(\vec{r}_x \times \vec{r}_y = (-f_x, -f_y, 1)\)
- Implicit: \(F(x, y, z) = 0 \Rightarrow \vec{r}_x \times \vec{r}_y = \frac{1}{F_z} (F_x, F_y, F_z)\).

Since \(\|\vec{r}_u \times \vec{r}_v\| \, du \, dv\) was the area of the parallelogram approximating \(\vec{F}(R; \psi)\), it makes sense to define the surface area of \(S\) by

\[
a(S) := \iint_S \|\vec{r}_u \times \vec{r}_v\| \, du \, dv
\]

in the parametric setting.

- Explicit setting: \(\iint_S \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)

**Ex 5** We find the surface area of the upper hemisphere of radius \(R\) in two ways:

- Parametrically: \(\iint_S R \sin v \|\vec{r}(u, v)\| \, du \, dv\)
  
  \[
  (S) = [0, 2\pi] \times [0, \frac{\pi}{2}] 
  \]
  
  \[
  R^2 \int_0^{2\pi} \int_0^{\frac{\pi}{2}} R^2 \sin v \, dv \, du 
  = 2\pi R^2.
  \]

- Implicitly: \(D = \text{disk of radius } R, \ F(x, y, z) = x^2 + y^2 + z^2 - R^2\)
  
  \[
  \iint_D \sqrt{(2x')^2 + (2y')^2 + (2z')^2} \, dx' \, dy' 
  = \iint_D \frac{2R}{z'} \, dx' \, dy' 
  \]
  
  \[
  = R \int_0^{2\pi} \int_0^{R} \frac{1}{\sqrt{R^2 - r^2}} \, r \, dr \, d\theta 
  \]
  
  Convert to polar: \(2\pi R \int_0^R (R^2 - r^2)^{-\frac{1}{2}} \, r \, dr\)
  
  \[
  = 2\pi R \left[-(R^2 - r^2)^{\frac{1}{2}} \right]_0^R 
  = 2\pi R^2.
  \]
Summary

- Parameter: \((x, y, z) = \vec{r}(u, v) = (x(u, v), y(u, v), z(u, v))\) normal to \(\vec{n}\)
  
  \(\vec{r}_u \times \vec{r}_v = -R \sin v \vec{r}(u, v)\) for sphere

- Explicit: \(z = f(x, y)\) \(\Rightarrow \vec{r}(x, y) = (x, y, f(x, y))\)
  \(\vec{r}_x \times \vec{r}_y = (-f_y, f_x, 1)\)

- Implicit: \(F(x, y, z) = 0\) \(\Rightarrow \vec{r}_x \times \vec{r}_y = \frac{1}{F_z}(F_x, F_y, F_z)\).

Ex 5

We find the surface area of the upper hemisphere of radius \(R\) in two ways:

- Parametrically:
  \[
  \iint_S R \sin v \|\vec{r}(u, v)\| \, du \, dv
  \]
  \(S = \{(x, y) \mid 0 \leq x^2 + y^2 \leq R^2\}\)
  \[
  = R^2 \int_0^{2\pi} \int_0^R \sin v \, dv \, du
  \]
  \[
  = 2\pi R^2.
  \]

- Implicitly:
  \(D = \) disk of radius \(R\), \(F(x, y, z) = x^2 + y^2 + z^2 - R^2\)
  \[
  \iint_D \frac{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}{2z} \, dx \, dy
  \]
  \[
  = R \int_0^{2\pi} \int_0^R \frac{1}{\sqrt{R^2 - r^2}} \, r \, dr \, d\theta
  \]
  \[
  \approx 2\pi R \int_0^R (R^2 - r^2)^{-\frac{1}{2}} \, r \, dr
  \]
  \[
  = 2\pi R \left[ -\left(\frac{R^2 - r^2}{2}\right) \right]_0^R = 2\pi R^2.
  \]

Technically, one should work with the part of \(D\) over a disk of radius \(R_0 < R\) and take \(\lim_{R_0 \to R}\) to avoid singularities of \(F(x, y, z)\).
**Summary**

- parameter: \((x, y, z) = \hat{r}(u, v) = F_x \hat{v}_x + F_y \hat{v}_y \) normal to \(S\)

  \( \hat{r}(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v) \)

  \[ \hat{r}_x \times \hat{r}_y = -R \sin v \hat{r}(u, v) \] for sphere

- explicit: \( z = F(x, y) \) then \( \hat{r}(x, y) = (x, y, F(x, y)) \)

  \[ \hat{r}_x \times \hat{r}_y = (-F_x, -F_y, 1) \]

- implicit: \( F(x, y, z) = 0 \) then \( \hat{r}_x \times \hat{r}_y = \frac{1}{F_z} (F_x, F_y, F_z) \).

---

**Ex 6**

Compute the surface area of the part of the sphere of radius 4 between \( z = -2 \) and \( z = 2 \).

Since \( \| \hat{r}_x \times \hat{r}_y \| \) is the area of the parallelogram approximating \( F(R, \hat{r}) \), it makes sense to define the surface area of \( S \) by

\[ \sigma(S) := \iint_S \| \hat{r}_x \times \hat{r}_y \| \, du \, dv \]

in the parametric setting.

- explicit setting: \( \iint_S \sqrt{1 + F_x^2 + F_y^2} \, dx \, dy \)

- implicit setting: \( \iint_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy \)
Summary

- Parameter: \((x, y, z) = \xi(u, v) = \vec{F}_x \times \vec{F}_y\) normal to \(S\)
  
  \[\vec{F}_x \times \vec{F}_y = -R \sin v \hat{n}(u, v)\] for sphere

- Explicit: \(z = F(x, y) = \vec{F}_z \cdot (x, y, f(x, y)) = \vec{F}_x \times \vec{F}_y = (-\vec{F}_x, -\vec{F}_y, 1)\)

- Implicit: \(F(x, y, z) = 0 \Rightarrow \vec{F}_x \times \vec{F}_y = \frac{1}{F_z} (\vec{F}_x, \vec{F}_y, \vec{F}_z)\).

Since \(\|\vec{F}_x \times \vec{F}_y\|\) was the area of the parallelogram approximating \(F(R, \theta)\), it makes sense to define the surface area of \(S\) by

\[a(S) := \iint_S \|\vec{F}_x \times \vec{F}_y\|\, du\, dv\]

in the parametric setting.

- Explicit setting: \(\iint_S \sqrt{1 + \vec{F}_x^2 + \vec{F}_y^2}\, dx\, dy\)

- Implicit setting: \(\iint_S \sqrt{\frac{\vec{F}_x^2 + \vec{F}_y^2 + \vec{F}_z^2}{|F_z|}}\, dx\, dy\)

Ex 6*

Compute the surface area of the part of the sphere of radius 4 between \(z = -2\) and \(z = 2\).

We must limit \(4 \cos v\) to between \(\pm 2\), i.e., \(\cos v \in \left[-\frac{1}{2}, \frac{1}{2}\right] \Rightarrow v \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\). Then

\[a(S) = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \int_{-2}^{2} R^2 \sin v \, du\, dv = 2\pi R^2 \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin v\, dv\]

\[= 32\pi \left[-\cos v\right]_{\frac{\pi}{3}}^{\frac{2\pi}{3}} = 32\pi .\]
**Summary**

- Parameter: \((x, y, z) = \hat{r}(u, v) = (F_x, F_y, F_z) \) normal to \(S\)
- \(F_x \times F_y\) for sphere
- \(F_x \times F_z = (-F_y, F_x, 0)\)
- \(F_y \times F_z = (F_x, -F_y, 0)\)
- \(\hat{r} \times \hat{r} = \frac{1}{F_z} (F_x, F_y, F_z)\)

Since \(\|F_x \times F_y\|\) was the area of the parallelogram approximating \(\hat{F}(R; \theta)\), it makes sense to define the surface area of \(S\) by

\[
a(S) := \int_S \|F_x \times F_y\| \, dudv\]

in the parametric setting.

- Explicit setting: \(\int_S \sqrt{1 + f_x^2 + f_y^2} \, dxdy\)
- Implicit setting: \(\int_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dxdy\)

**Ex 6**

Compute the surface area of the part of the sphere of radius 4 between \(z = -2\) and \(z = 2\).

We must limit \(4 \cos v (\cos u)\) to between \(\pm 2\), i.e. \(\cos v \in \left[-\frac{1}{2}, \frac{1}{2}\right] \Rightarrow v \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\). Then

\[
a(u) = \int_{\pi/3}^{2\pi/3} \int_0^{2\pi/3} R^2 \sin v \, du \, dv = 2\pi R^2 \int_{\pi/3}^{2\pi/3} \sin v \, dv = 32\pi \left[-\cos v\right]_{\pi/3}^{2\pi/3} = 32\pi.
\]

**Ex 7**

Find the area of the part of the plane \(x + 2y + 3z = 1\) that lies inside the cylinder \(x^2 + y^2 = 3\).
**Summary**

- Parameter: \((x, y, z) = \Phi(u, v) = \frac{F_x \times F_y}{F_z} \) normal to \(S\)
- \(\Phi(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\)
  
- Explicit: \(z = F(x, y) = \Phi(x, y) = (x, y, f(x, y))\)
  
- \(\Phi_x \times \Phi_y = (-f_y, -f_z, 1)\)
- Implicit: \(F(x, y, z) = 0 \Rightarrow \Phi_x \times \Phi_y = \frac{1}{F_z} (F_x, F_y, F_z)\).

Since \(\left\| \Phi_x \times \Phi_y \right\| \) was the area of the parallelogram approximating \(\Phi(Rij)\), it makes sense to define the surface area of \(S\) by

\[
a(S) := \iint_S \left\| \Phi_x \times \Phi_y \right\| \, du \, dv
\]

in the parameter setting.

- Explicit setting: \(\iint_S \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)

**Ex 6**

Compute the surface area of the part of the sphere of radius 4 between \(z = -2\) and \(z = 2\).

We must limit \(4 \cos v (=z)\) to between \(\pm 2\), i.e. \(\cos v \in [-\frac{1}{2}, \frac{1}{2}] \Rightarrow v \in [\frac{\pi}{3}, \frac{2\pi}{3}]\). Then

\[
a(A) = \int_{\pi/3}^{2\pi/3} \int_0^{2\pi} R^2 \sin v \, du \, dv = 2\pi R^2 \int_{\pi/3}^{2\pi/3} \sin v \, dv
\]

\[
= 32\pi \left[-\cos v \right]_{\pi/3}^{2\pi/3} = 32\pi.
\]

**Ex 7**

Find the area of the part of the plane \(x + 2y + 3z = 1\) that lies inside the cylinder \(x^2 + y^2 = 3\). Use \(z = f(x, y) = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y\)

\[
a(A) = \iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy
\]

\[
= \frac{\sqrt{14}}{3} a(A) = \frac{\sqrt{14}}{3} \pi (15)^2 = \pi \sqrt{14}.
\]
**Summary**

- Parameter: \((x,y,z) = \vec{r}(u,v) = F_x \times F_v\) normal to \(S\)
  
  \( F_x \times F_v = -R \sin v \vec{r}(u,v) \) for sphere

- Explicit: \(z = f(x,y) \Rightarrow \vec{r}(x,y) = (x, y, f(x,y))\)
  
  \[ \vec{r}_x \times \vec{r}_y = (F_x, F_y, F_z) \]

- Implicit: \(F(x,y,z) = 0 \Rightarrow \vec{r}_x \times \vec{r}_y = \frac{1}{|F|} (F_x, F_y, F_z)\).

Since \(\|\vec{r}_x \times \vec{r}_y\| du dv\) was the area of the parallelogram approximating \(F(R,\theta)\), it makes sense to define the **surface area of \(S\)** by

\[ a(S) = \iint_S \|\vec{r}_x \times \vec{r}_y\| \, du dv \]

in the parameter setting.

- Explicit setting: \(\iint_S \sqrt{1 + F_x^2 + F_y^2} \, dx \, dy\)

- Implicit setting: \(\iint_S \sqrt{F_x^2 + F_y^2 + F_z^2} / |F| \, dx \, dy\)

**Ex 6**

- Compute the surface area of the part of the sphere of radius 4 between \(z = -2\) and \(z = 2\).

We must limit \(4 \cos v (\cos \theta)\) to values \(-2\), i.e., \(\cos v \in [-\frac{1}{2}, \frac{1}{2}] \Rightarrow v \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]\). Then

\[ a(S) = \iiint_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} R^2 \sin v \, du \, dv = 2\pi R^2 \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \sin v \, dv \]

\[ = 32\pi \left[ -\cos v \right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = 32\pi. \]

**Ex 7**

- Find the area of the part of the cylinder \(x^2 + y^2 = 3\) that lies inside the cylinder \(x^2 + y^2 = 3\).

Use \(z = f(x,y) = \frac{1}{2} - \frac{1}{3}x - \frac{1}{3}y\).

\[ a(S) = \iint_S \sqrt{1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} \, dx \, dy \]

\[ = \frac{\sqrt{14}}{3} a(S) = \sqrt{14} \pi (15)^2 = \pi \sqrt{14}. \]

Note that \(\cos (\theta) = (\vec{r}_x \times \vec{r}_y) \cdot \vec{e}_z = \frac{1}{\|\vec{r}_x \times \vec{r}_y\|} \|\vec{r}_x \times \vec{r}_y\| \vec{e}_z \]

\[ \Rightarrow \frac{\sqrt{14}}{3} = \frac{1}{\cos (\alpha)} \Rightarrow a(S) = \cos (\alpha) a(S). \]
**Summary**

- Parameter: \((x, y, z) = \vec{r}(u, v) = \vec{R}_u \times \vec{R}_v\) normal to \(S\)

  e.g. \(\vec{r}(u, v) = (R \cos u \sin v, R \sin u \sin v, R \cos v)\)
  \(\vec{r}_u \times \vec{r}_v = -R \sin v \vec{r}(u, v)\) for sphere.

- Explicit: \(z = f(x, y)\) \(\Rightarrow\) \(\vec{r}(x, y) = (x, y, f(x, y))\)
  \(\vec{r}_x \times \vec{r}_y = (f_y, -f_x, 1)\)

- Implicit: \(F(x, y, z) = 0\) \(\Rightarrow\) \(\vec{r}_x \times \vec{r}_y = \frac{1}{F_z} (F_x, F_y, F_z)\).

Since \(\|\vec{r}_x \times \vec{r}_y\| du dv\) was the area of the parallelogram approximating \(F(R_x, R_y)\), it makes sense to define the surface area of \(S\) by

\[
a(S) := \iint_S \|\vec{r}_x \times \vec{r}_y\| du dv
\]

in the parametric setting.

- Explicit setting: \(\iint_S \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)

---

**Ex 8**

The length of the curve shown is

\[ l = \int_a^b \sqrt{1 + f'(s)^2} \, ds. \]

Its centroid has \(u\)-coordinate

\[ \bar{u} = \int_a^b u \sqrt{1 + f'(s)^2} \, ds / l. \]
**Summary**
- Parameter: \((x,y,z) = \mathbf{r}(u,v) = F_u \times F_v\) normal to \(S\)
  - \(\mathbf{r}(u,v) = (R\cos u \sin v, R\sin u \sin v, R\cos v)\)
  - \(\mathbf{r}_u \times \mathbf{r}_v = -R\sin v \mathbf{r}(u,v)\) for sphere
- Explicit: \(z = f(x,y)\) \(\Rightarrow \mathbf{r}(x,y) = (x,y,f(x,y))\)
  - \(\mathbf{r}_u \times \mathbf{r}_v = (-f_x, -f_y, 1)\)
- Implicit: \(F(x,y,z) = 0\) \(\Rightarrow \mathbf{r}_u \times \mathbf{r}_v = \frac{1}{|F_z|} (F_x, F_y, F_z)\).

Since \(\|\mathbf{r}_u \times \mathbf{r}_v\|\) was the area of the parallelogram approximating \(F(R;\mathbf{j})\), it makes sense to define the surface area of \(S\) by
\[
\sigma(S) := \iint_S \|\mathbf{r}_u \times \mathbf{r}_v\| \, du \, dv
\]
in the parametric setting.

- Explicit setting: \(\iint_S \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_S \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)

**Ex 8**
The length of the curve shown is
\[
l = \int_a^b \sqrt{1 + f'(s)^2} \, ds.
\]
Its centroid has \(u\)-coordinate
\[
\overline{u} = \frac{1}{l} \int_a^b u \sqrt{1 + f'^2} \, ds / l.
\]
The surface of revolution obtained by revolving about the vertical is parameterized by
\[
\mathbf{r}(u,v) = (u \cos v, u \sin v, f(u)), \quad (u,v) \in [a,b] \times [0,\pi]
\]
\(\Rightarrow \mathbf{r}_u = (\cos v, \sin v, f'(u)), \mathbf{r}_v = (-u \sin v, u \cos v, 0)\)
**Summary**

- Parameter: \((x, y, z) = \bar{r}(u, v) = F_x \times F_y \) normal to \(S\)
  - \(F_x \times F_y = (R \cos u \sin v, R \sin u \sin v, R \cos v)\) 
  - \(\bar{r}_u \times \bar{r}_v = R \sin v \bar{r}(u, v)\) for sphere
- Explicit: \(z = f(x, y) \Rightarrow \bar{r}(x, y) = (x, y, f(x, y))\)
  - \(\bar{r}_x \times \bar{r}_y = (-f_y, -f_x, 1)\)
- Implicit: \(F(x, y, z) = 0 \Rightarrow \bar{r}_x \times \bar{r}_y = \frac{1}{F_z} \) \((F_x, F_y, F_z)\).

Since \(\|\bar{r}_u \times \bar{r}_v\| \) was the area of the parallelogram approximating \(\bar{F}(R; \gamma)\), it makes sense to define the surface area of \(S\) by

\[
a(S) := \iint_S \|\bar{r}_u \times \bar{r}_v\| \, du \, dv
\]

in the parametric setting.

- Explicit setting: \(\iint_D \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy\)
- Implicit setting: \(\iint_D \sqrt{F_x^2 + F_y^2 + F_z^2} \, \frac{1}{|F_z|} \, dx \, dy\)

**Example 8**

The length of the curve shown is

\[
l = \int_a^b \sqrt{1 + f'(u)^2} \, du.
\]

Its curvature has \(u\)-coordinate

\[
u = \int_a^b u \sqrt{1 + f''(u)^2} \, du \Bigg/ l.
\]

The surface of revolution obtained by revolving about the vertical is parameterized by

\[
\bar{r}(u, v) = (u \cos v, u \sin v, f(u)), \quad (u, v) \in [a, b] \times [0, \pi]
\]

\[
\Rightarrow \bar{r}_u = (\cos v, \sin v, f'(u)), \quad \bar{r}_v = (-u \sin v, u \cos v, 0)
\]

\[
\Rightarrow \bar{r}_u \times \bar{r}_v = u (-f'(u) \cos v, -f'(u) \sin v, 1)
\]

\[
\Rightarrow \|\bar{r}_u \times \bar{r}_v\| = u \sqrt{1 + f'(u)^2}.
\]
SUMMARY

- parameter: \((x,y,z) = \mathbf{F}(u,v) \Rightarrow \mathbf{F}_u \times \mathbf{F}_v\) normal to \(S\)
  
  \[ \mathbf{F}_u = \left( \frac{x_u}{2}, \frac{y_u}{2}, \frac{z_u}{2} \right) \]
  
- explicit setting: \(z = F(x,y) \Rightarrow \mathbf{F}(x,y) = (x, y, f(x,y))\)
  \[ \mathbf{F}_u \times \mathbf{F}_v = (-f_u, -f_v, 1) \]

- implicit setting: \(FG(x,y,z) = 0 \Rightarrow \mathbf{F}_u \times \mathbf{F}_v = \frac{1}{F_z} (F_x, F_y, F_z). \)

Since \(\|\mathbf{F}_u \times \mathbf{F}_v\|\) was the area of the parallelogram approximating \(F(R_{ij})\), it makes sense to define the surface area of \(S\) by

\[ a(S) := \int \int_{R_{ij}} \|\mathbf{F}_u \times \mathbf{F}_v\| \, du \, dv \]

in the parametric setting.

- explicit setting: \(\int \int_S \sqrt{1 + F_x^2 + F_y^2} \, dx \, dy\)

- implicit setting: \(\int \int_{R_{ij}} \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy\)

\[ \mathbf{Ex 8}\] The length of the curve shown is

\[ l = \int_a^b \sqrt{1 + (f'(u))^2} \, du. \]

Its centroid has \(u\)-coordinate

\[ \bar{u} = \int_a^b u \sqrt{1 + (f'(u))^2} \, du \bigg/ l. \]

The surface of revolution obtained by revolving about the vertical is parameterized by

\[ \mathbf{F}(u,v) = (u \cos v, u \sin v, f(u)) \]

\[ \mathbf{F}_u = (\cos v, \sin v, f'(u)), \mathbf{F}_v = (-u \sin v, u \cos v, 0) \]

\[ \Rightarrow \mathbf{F}_u \cdot \mathbf{F}_v = u (-f'(u) \cos v, -f'(u) \sin v, 1) \]

\[ \Rightarrow \|\mathbf{F}_u \times \mathbf{F}_v\| = u \sqrt{1 + (f'(u))^2}. \]

So

\[ a(S) = \int_0^{2\pi} \int_0^b u \sqrt{1 + (f'(u))^2} \, du \, dv \]

\[ = 2\pi \bar{u} l. \]

Length of curve

\[ \text{circumference} \]

\[ \text{traced out by} \]

\[ \text{centroid} \]