Lecture 44: Surface area

For most of the remainder of this course we shall work in $\mathbb{R}^3$.

We know how to integrate on

- curves in $\mathbb{R}^3$ &
- solid regions in $\mathbb{R}^3$;

what's left is

- surfaces in $\mathbb{R}^3$.

Here are 3 ways to describe a surface:

$\longrightarrow$ <u>implicitly</u>: as level surface $F(x,y,z) = 0$

$\longrightarrow$ <u>explicitly</u>: as graph $z = f(x,y)$ (or $y = f(x,z)$, etc.)

$\longrightarrow$ <u>parametrically</u>: $(u,v) \longmapsto (X(u,v), Y(u,v), Z(u,v)) =: \vec{F}(u,v)$
where $(u,v) \in T \subset \mathbb{R}^2$.

Ex 1/ Cylinder of radius R & height h:

<u>implicit</u>: $x^2+y^2 = R^2$

<u>explicit</u>: $y = \pm\sqrt{R^2-x^2}$

and $0 \le z \le h$

<u>parametric</u>: $\vec{r}(u,v) = (R\cos u, R\sin u, v)$
$(u,v) \in [0,2\pi] \times [0,h]$

//

Ex 2/ Sphere of radius R:

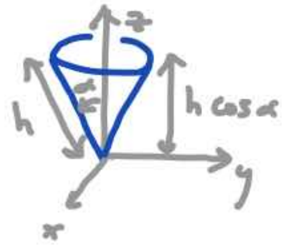
<u>implicit</u>: $x^2+y^2+z^2 = R^2$

<u>explicit</u>: $z = \pm\sqrt{R^2-x^2-y^2}$

<u>parametric</u>: $\vec{r}(u,v) = (R\cos u\sin v, R\sin u\sin v, R\cos v)$
$(u,v) \in [0,2\pi] \times [0,\pi]$

//

Ex 3/ Cone of vertical angle α and "height" h:



implicit: $x^2+y^2 = z^2\tan(\alpha)$ $\left(\frac{r}{z} = \tan\alpha\right)$

explicit: $z = \cot(\alpha)\sqrt{x^2+y^2}$

parametric:

$\vec{r}(u,v) = (v\sin\alpha\cos u, v\sin\alpha\sin u, v\cos\alpha)$ //

To be more explicit about what sorts of surfaces we'll work with:

Let

- $\mathcal{T} \subset \mathbb{R}^2$ (coords. u,v) be a region w/piecewise smooth boundary $\partial\mathcal{T}$
- $\vec{r}: \mathcal{T} \to \mathbb{R}^3$ be a 1-to-1, C^1 map (technically, we want this to extend to an open set containing $\mathcal{T}$)
- $\mathcal{S} := \vec{r}(\mathcal{T})$ the image of $\vec{r}$.

Definition: $\vec{r}$ is a smooth parametrization of $\mathcal{S}$ if $\vec{r}_u \times \vec{r}_v$ is never zero. $\mathcal{S}$ is a smooth surface if it has a smooth parametrization (or a union of "open subsets" with such parametrizations).

[For instance, one can show that $\mathcal{S} = \{(x,y,z) \mid F(x,y,z) = 0\}$ is smooth if, at every point on it, F has a nonvanishing partial derivative ("Implicit Function Theorem").]

Notes: write $\vec{r}(u,v) = (X(u,v), Y(u,v), Z(u,v))$

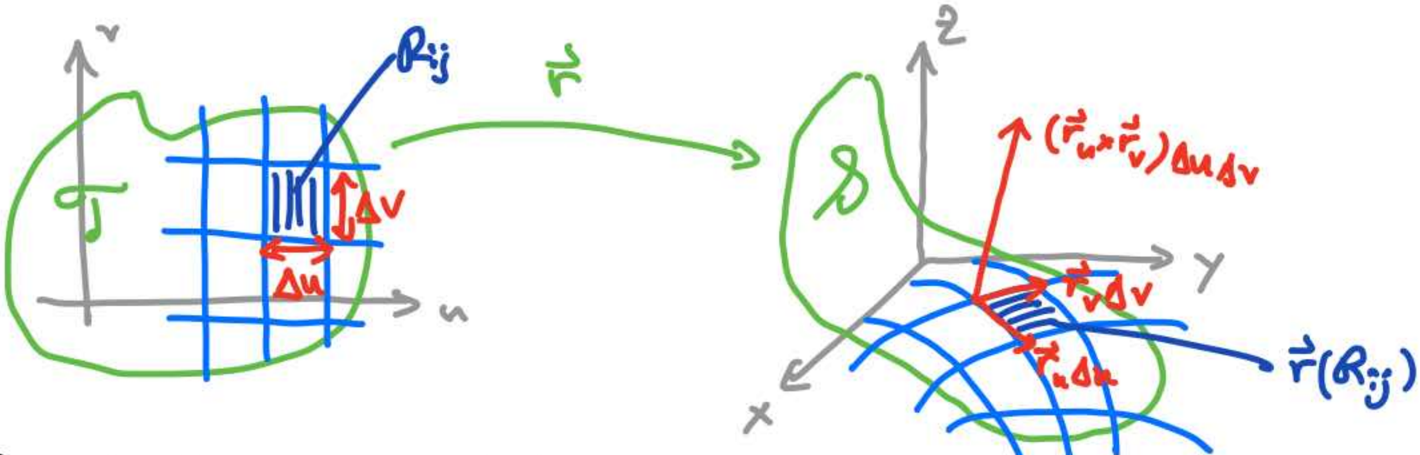
① The condition in the Definition is the same as asking

$$J_{\vec{r}} = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix}$$

$\vec{r}_u$ $\vec{r}_v$

to have rank 2 everywhere. (Why?)

② $\vec{r}_u \times \vec{r}_v$ should be viewed as a vector normal to $\mathcal{S}$, with length equal to the ratio of areas $a(\vec{r}(R_{ij}))/a(R_{ij})$:



③ If $\mathcal{S}$ is described "explicitly" by $z = f(x,y)$, $(x,y) \in \mathcal{D} \subset$ xy-plane, then it is described parametrically by $\vec{r}(x,y) = (x, y, f(x,y))$.

[So $(u,v) = (x,y)$, $\mathcal{T} = \mathcal{D}$]

Ex 4 / For the sphere, with $\vec{r}(u,v) = (R\cos u \sin v, R \sin u \sin v, R\cos v)$,

we have $\vec{r}_u = (-R\sin u \sin v, R\cos u \sin v, 0)$,

$\vec{r}_v = (R\cos u \sin v, R\sin u \cos v, -R\sin v)$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = (-R^2 \cos u \sin^2 v, -R^2 \sin u \sin^2 v, -R^2 \sin v \cos v)$$

$\|\vec{r}(u,v)\| = R$ on sphere!

$$= -R\sin(v)\,\vec{r}(u,v).$$

$$\Rightarrow \|\vec{r}_u \times \vec{r}_v\| = R^2 \sin(v).$$

Note that this is singular at $v = 0$ and π (the north & south poles), and not 1-to-1 at $u = 0$ & 2π. But because this happens on a set of content zero in $\mathcal{T} = [0, 2\pi] \times [0, \pi]$, it isn't a problem for using the parametrization $\vec{r}$ to integrate over $\mathcal{S}$. //

A bit more on ② & ③:

- one way to think about $\vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$ being $\perp$ to $\mathcal{S}$ at $\vec{r}(u_0,v_0)$ is by showing it is $\perp$ to the tangent vector of any curve $\mathcal{C} \subset \mathcal{S}$ passing through this point:

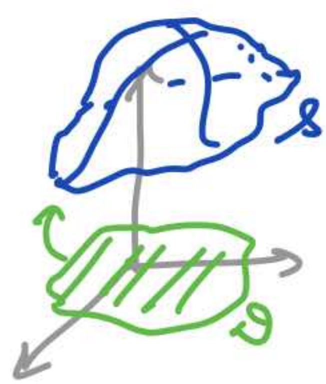
view $\mathcal{C} = \vec{r}(\mathcal{C}^*)$ for $\mathcal{C}^* \subset \mathcal{T}$ parametrized by $\vec{\alpha}(t)$, so that $\vec{r} \circ \vec{\alpha}$ parametrizes $\mathcal{C}$. Then $\vec{\alpha}'(t) = \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix}$ and $(\vec{r} \circ \vec{\alpha})'(t) = \begin{pmatrix} \uparrow & \uparrow \\ \vec{r}_u & \vec{r}_v \\ \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = u'(t)\vec{r}_u + v'(t)\vec{r}_v$ is $\perp$ to $\vec{r}_u \times \vec{r}_v$.

- In ③, $\vec{r}_u = \vec{r}_x = (1, 0, f_x)$ & $\vec{r}_v = \vec{r}_y = (0, 1, f_y)$ $\Longrightarrow \vec{r}_u \times \vec{r}_v = (-f_x, -f_y, 1)$.
- If $f(x,y)$ is defined "implicitly" by $F(x, y, f(x,y)) = 0$, then recall that applying $\frac{\partial}{\partial x} \Rightarrow F_x + f_x F_z = 0$ $\Longrightarrow f_x = -F_x/F_z$ etc. $\Rightarrow \vec{r}_u \times \vec{r}_v = \left(\frac{F_x}{F_z}, \frac{F_y}{F_z}, 1\right) = \frac{1}{F_z}\vec{\nabla}F$.

Since $\|\vec{r}_u \times \vec{r}_v\| \Delta u \Delta v$ was the area of the parallelogram approximating $\vec{r}(R_{ij})$, it makes sense to define the surface area of $\mathcal{S}$ by

$$a(\mathcal{S}) := \iint_{\mathcal{T}} \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

in the parametric setting.



- explicit setting: $\displaystyle\iint_{D} \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy$
- implicit setting: $\displaystyle\iint_{D} \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|} \, dx \, dy$

Ex 5/ Find the surface area of the upper hemisphere of radius R.

- parametric approach: $\displaystyle\iint_{\underset{[0,2\pi]\times[0,\pi/2]}{\mathcal{T}}} R^2 \sin v \, du \, dv = R^2 \int_0^{2\pi} \underbrace{\int_0^{\pi/2} \sin v \, dv}_{=1} \, du = 2\pi R^2$

- <u>implicit approach</u>: $\mathcal{D}$ = disk of radius R, $F(x,y,z) = x^2+y^2+z^2-R^2$

$$\iint_{\mathcal{D}} \frac{\sqrt{(2x)^2+(2y)^2+(2z)^2}}{2z}\,dx\,dy = \iint_{\mathcal{D}} \frac{\cancel{2}R}{\cancel{2}z}\,dx\,dy \qquad z = \sqrt{R^2-x^2-y^2} = \sqrt{R^2-r^2}$$

convert to polar

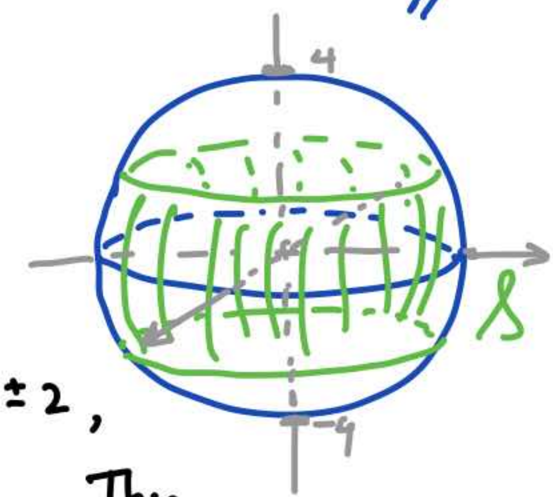
$$= R\int_0^{2\pi}\int_0^R \frac{1}{\sqrt{R^2-r^2}}\,r\,dr\,d\theta = 2\pi R\int_0^R (R^2-r^2)^{-1/2}\,r\,dr$$

$$= 2\pi R\left[-(R^2-r^2)^{1/2}\right]_0^R = 2\pi R^2.$$

Technically, one should work with the part of $\mathcal{S}$ over a disk of radius $R_0 < R$ and take $\lim_{R_0 \to R}$, to avoid singularities of $\vec{r}(x,y)$.

//

Ex 6 / Compute the surface area of the part of the sphere of radius $R=4$ between $z=-2$ & $z=2$.



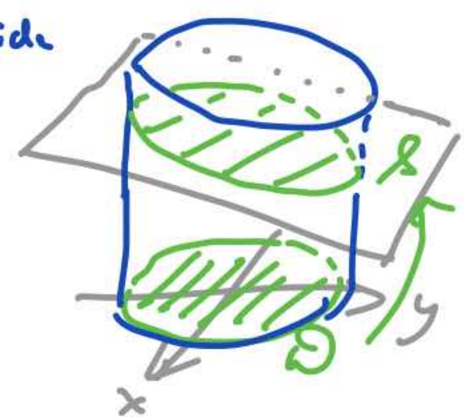
We must limit $4\cos v$ $(=z)$ to between ± 2, i.e. $\cos v \in \left[-\frac{1}{2}, \frac{1}{2}\right] \Rightarrow v \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$. Then

$$a(\mathcal{S}) = \int_{\pi/3}^{2\pi/3}\int_0^{2\pi} R^2 \sin v\,du\,dv = 2\pi R^2 \int_{\pi/3}^{2\pi/3} \sin v\,dv$$

$$= 32\pi\left[-\cos v\right]_{\pi/3}^{2\pi/3} = 32\pi.$$

//

Ex 7 / Find the area of the part of the plane $x+2y+3z=1$ that lies inside the cylinder $x^2+y^2=3$.



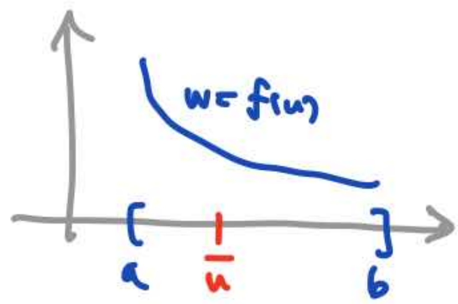
Use $z = f(x,y) = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$

$$a(\mathcal{S}) = \iint_{\mathcal{D}} \sqrt{1+\left(-\tfrac{1}{3}\right)^2+\left(-\tfrac{2}{3}\right)^2}\, dx\, dy = \frac{\sqrt{14}}{3}\, a(\mathcal{D})$$

$$= \frac{\sqrt{14}}{3}\,\pi(\sqrt{3})^2 = \pi\sqrt{14}.$$

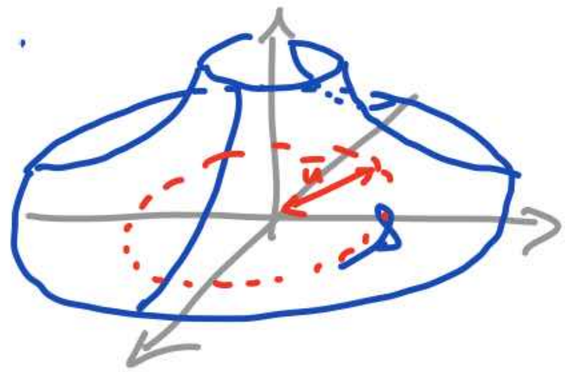
//

Ex 8 / The length of the curve shown is $\ell = \int_a^b \underbrace{\sqrt{1+f'(u)^2}\, du}_{ds}$



Its centroid has u-coordinate

$$\bar{u} = \int_a^b u \underbrace{\sqrt{1+f'(u)^2}\, du}_{ds} \Big/ \ell .$$



Find the area of the surface of revolution shown.

Parametrize $\mathcal{S}$ by

$$\vec{r}(u,v) = (u\cos v,\ u\sin v,\ f(u)), \quad (u,v) \in [a,b] \times [0,2\pi]$$

$$\Rightarrow \vec{r}_u = (\cos v,\ \sin v,\ f'(u)), \quad \vec{r}_v = (-u\sin v,\ u\cos v,\ 0)$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = u\left(-f'(u)\cos v,\ -f'(u)\sin v,\ 1\right)$$

$$\Rightarrow \|\vec{r}_u \times \vec{r}_v\| = u\sqrt{1+(f'(u))^2}$$

$$\Rightarrow a(\mathcal{S}) = \int_0^{2\pi}\int_a^b u\sqrt{1+f'(u)^2}\, du\, dv$$

$$= \underbrace{2\pi\bar{u}}_{\text{circumference traced out by centroid}}\ \underbrace{\ell}_{\text{arclength of curve}}.$$

//