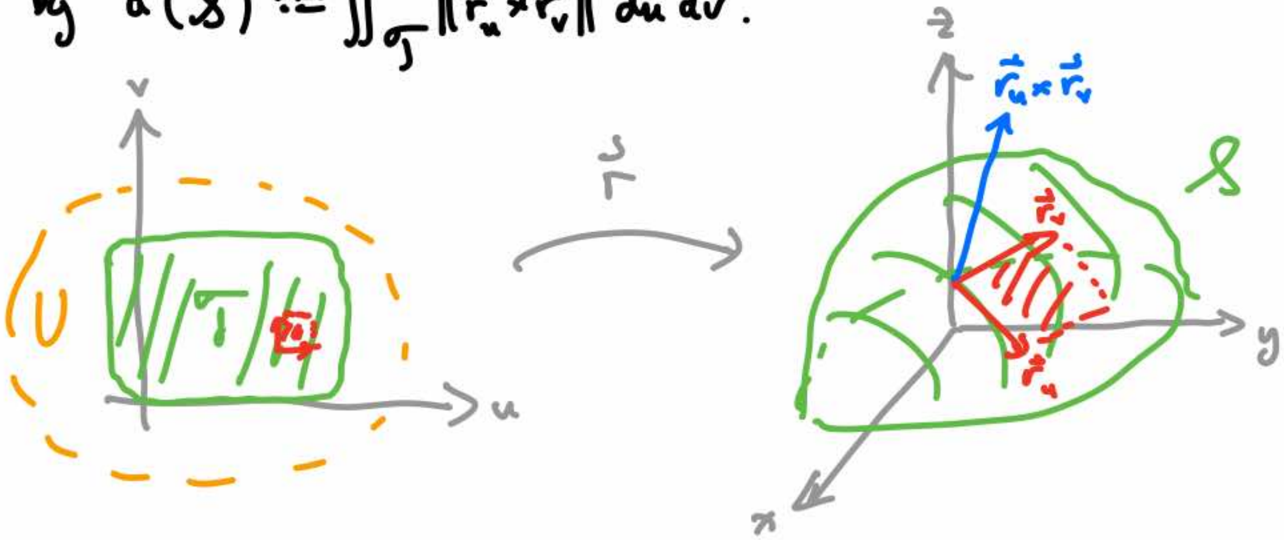
Lecture 45: Surface Integrals

Let's briefly review the definition of surface area from last time: let

- $\mathcal{T} \subset \mathbb{R}^2$ be a region with piecewise-smooth boundary $\partial\mathcal{T}$
- $\vec{r} : U \to \mathbb{R}^3$ be a 1-to-1, C^1 map on an open set $U \subset \mathbb{R}^2$ containing $\mathcal{T}$
- $\mathcal{S} := \vec{r}(\mathcal{T})$ be the image of $\mathcal{T}$ under $\vec{r}$
- assume that $\vec{r}_u \times \vec{r}_v$ is everywhere nonzero;
  $\hat{n} := \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ is the unit normal to the parametric surface $\mathcal{S}$
- since $\|\vec{r}_u \times \vec{r}_v\|$ is the dilation factor on areas, define surface area by $a(\mathcal{S}) := \iint_{\mathcal{T}} \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$.



Now let $g : \mathcal{S} \to \mathbb{R}$ be a bounded function on $\mathcal{S}$ (or some domain containing $\mathcal{S}$). The <u>surface integral</u> (of g over $\mathcal{S}$) is defined by

$$\iint_{\mathcal{S}} g \, dS := \iint_{\mathcal{T}} g(\vec{r}(u,v)) \, \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

whenever the right-hand integral exists. This recovers surface area when $g \equiv 1$. If g is mass density, then the center-of mass coordinates are $\bar{x} = \iint_{\mathcal{S}} x g \, d\mathcal{S} \Big/ \underbrace{\iint_{\mathcal{S}} g \, d\mathcal{S}}_{\text{total mass}}$ / etc.
(= coords. of centroid if $g \equiv 1$).

Ex 1/ Find the centroid of a hemisphere of radius R, using $\vec{r}(u,v) = (R\cos u \sin v, R \sin u \sin v, R\cos v)$.

By symmetry $\bar{x} = \bar{y} = 0$. We know $a(\mathcal{S}) = 2\pi R^2$. So

$$\bar{z} = \frac{1}{2\pi R^2} \iint_{\mathcal{S}} z \, d\mathcal{S} = \frac{1}{2\pi R^2} \int_0^{\pi/2} \int_0^{2\pi} \underbrace{(R\cos v)}_{z} \underbrace{R^2 \sin v}_{\|\vec{r}_u \times \vec{r}_v\|} \, du \, dv$$

$$= \frac{R^3}{2\pi R^2} \int_0^{2\pi} du \int_0^{\pi/2} \cos v \sin v \, dv = R \left[\tfrac{1}{2} \sin^2 v \right]_0^{\pi/2} = \frac{R}{2}. \quad //$$

The definition of surface integral is analogous to the line integral with respect to arclength on a curve, where given $\vec{r} : [a,b] \to \mathcal{C}$ and $h : \mathcal{C} \to \mathbb{R}$ we put $\int_{\mathcal{C}} h \, ds := \int_a^b h(\vec{r}(t)) \|\vec{r}\,'(t)\| \, dt$.
Is there an analogue of $\int_{\mathcal{C}} \vec{F} \cdot d\vec{r}$?

... well, not quite, because $d\vec{r} = \vec{r}\,'(t)\,dt$ is the tangent direction to the curve, and there is more than one tangent direction to $\mathcal{S}$ at each point (namely, $\vec{r}_u$ & $\vec{r}_v$). But there is still a unique normal direction, so we can generalize $\int_{\mathcal{C}} \vec{F} \cdot \hat{n} \, ds$ to $\iint_{\mathcal{S}} \vec{F} \cdot \hat{n} \, d\mathcal{S}$, where $\vec{F} : \mathcal{S} \to \mathbb{R}^3$ is a vector field.

This is the flux of $\vec{F}$ across $\mathcal{S}$:

$$\text{Flux} := \iint_{\mathcal{S}} \vec{F}\cdot\hat{n}\, dS = \iint_{T} \vec{F}(\vec{r}(u,v))\cdot \overbrace{\frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}}^{\hat{n}} \|\vec{r}_u\times\vec{r}_v\|\, du\, dv$$

$$= \iint_{T} \underbrace{\vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)}_{\text{scalar triple-product!}}\, du\, dv$$

Ex 2/ Find the outward flux of $\vec{F}(x,y,z) := (x,y,z)$ through the unit sphere $\{x^2+y^2+z^2=1\} = \mathcal{S}$.

swapped order (why?)

$$\underbrace{\vec{F}(\vec{r}(u,v))}_{=\vec{r}(u,v) \text{ in this case}}\cdot(\vec{r}_v\times\vec{r}_u) = \begin{array}{c} \begin{vmatrix} \cos u \sin v & \sin u \sin v & \cos v \\ \cos u \cos v & \sin u \cos v & -\sin v \\ -\sin u \sin v & \cos u \sin v & 0 \end{vmatrix} \end{array} \begin{array}{l} \vec{r} \\ \vec{r}_v \\ \vec{r}_u \end{array}$$

$$= \cos v\,(\sin v \cos v) + \sin v\,(\sin^2 v) = \sin v.$$

So $\text{Flux} = \int_0^{\pi}\int_0^{2\pi} (\sin v)\, du\, dv = 2\pi\left[-\cos v\right]_0^{\pi} = 4\pi.$ //

Interpretation: Suppose we have on $\mathbb{R}^3$

- $\vec{V}$ = fluid velocity field $\quad$ m/sec
- ρ = fluid density function $\quad$ kg/m^3
- $\vec{F} = \rho\vec{V}$ = flux density $\quad \dfrac{\text{kg/sec}}{\text{m}^2}$

Then $\vec{F}\cdot\hat{n}$ is the flux density in the normal direction, and the flux through $\mathcal{S}$

- $\iint_{\mathcal{S}} \vec{F}\cdot\hat{n}\, dS$ is measured in kg/sec.

Next we discuss the surface analogue of $\int_{\mathcal{C}} P\,dx + Q\,dy$.

Recall that, given $\vec{r}:[a,b] \to \mathcal{C}$, this was defined by

$\int_a^b P(\vec{r}(t))\,x'(t)\,dt + \int_a^b Q(\vec{r}(t))\,y'(t)\,dt$, where $\vec{r}(t) = (x(t), y(t))$.

For a function P on $\mathcal{S}$, define

$$\iint_{\mathcal{S}} P(x,y,z)\, dx \wedge dy := \iint_{\mathcal{T}} P(\vec{r}(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv, \quad \text{where}$$

$\frac{\partial(x,y)}{\partial(u,v)} := \det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$ may be regarded as the Jacobian of

the map $\mathcal{T} \xrightarrow{\vec{r}} \mathbb{R}^3 \xrightarrow[(x,y)]{\text{project to}} \mathbb{R}^2$. The integrals

$\iint_{\mathcal{S}} P\, dy \wedge dz$, $\iint_{\mathcal{S}} P\, dx \wedge dz$ are defined in exactly the same way.

Note that the order of differentials matters:

$$\iint_{\mathcal{S}} P\, dy \wedge dx = \iint_{\mathcal{T}} P(\vec{r}(u,v)) \frac{\partial(y,x)}{\partial(u,v)}\, du\, dv \qquad \begin{vmatrix} y_u & y_v \\ x_u & x_v \end{vmatrix} = - \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

$$= -\iint_{\mathcal{S}} P\, dx \wedge dy.$$

We write $dy \wedge dx = -dx \wedge dy$ formally to emphasize this.

The flux can be expressed in these terms: if

$\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is the vector field on $\mathcal{S}$,

then

$$\iint_{S} \vec{F}\cdot\hat{n}\, dS = \iint_{T} (P,Q,R)\cdot(\vec{r}_u\times\vec{r}_v)\, du\, dv$$

$$= \iint_{T} \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\, dv$$

$$= \iint_{T} P\,\frac{\partial(y,z)}{\partial(u,v)}\, du\, dv + \iint_{T} Q\,\frac{\partial(z,x)}{\partial(u,v)}\, du\, dv + \iint_{T} R\,\frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

$$= \iint_{S} P\, dy\wedge dz + \iint_{S} Q\, dz\wedge dx + \iint_{S} R\, dx\wedge dy$$

is an analogue of $\int_{C}\vec{F}\cdot d\vec{r} = \int_a^b (P,Q)\cdot(x',y')\,dt = \int_{C} P\,dx + Q\,dy$,

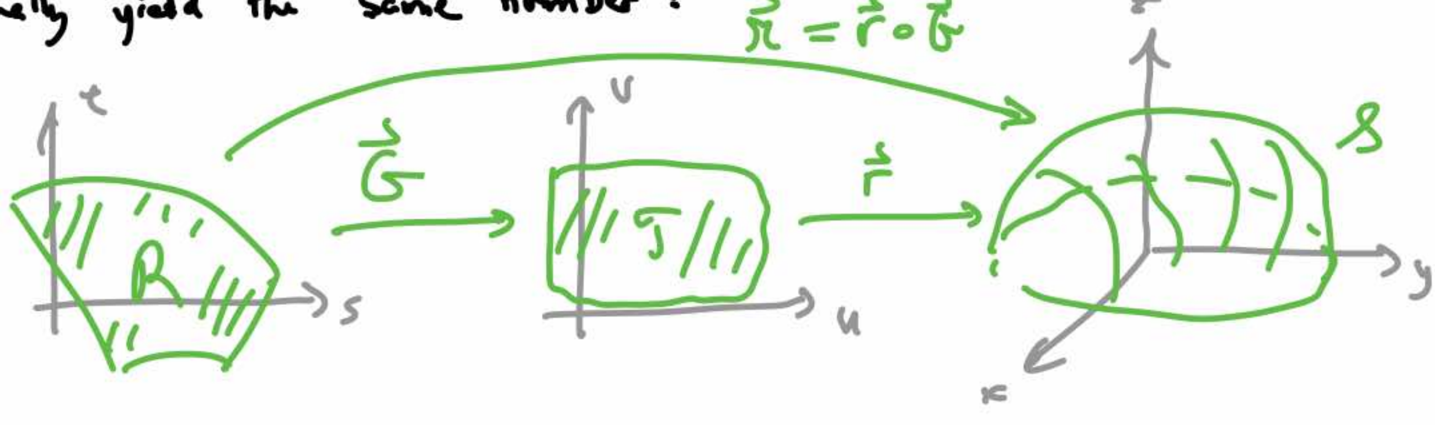
and will be helpful when discussing Stokes tomorrow.

Remark:

Returning to the basic integral $\iint_{S} g\, dS$ from which all of this took off, we need to check that it is well-defined — i.e., independent of the choice of parametrization. For if $\vec{R}: R \to S$ is another one, how do we know that

$$\iint_{T} g(\vec{r}(u,v))\,\|\vec{r}_u\times\vec{r}_v\|\, du\, dv \quad \text{and} \quad \iint_{R} g(\vec{R}(s,t))\,\|\vec{R}_s\times\vec{R}_t\|\, ds\, dt$$

actually yield the same number?



Well, the change-of-variable formula says that

$$\iint_{\mathcal{S}} (g\circ\vec r)(u,v)\,\|\vec r_u\times\vec r_v\|\,du\,dv = \iint_R (g\circ\vec r)(\vec G(s,t))\,\|\vec r_u(\vec G(s,t))\times\vec r_v(\vec G(s,t))\|\,|\det J_{\vec G}|\,ds\,dt.$$

so need to show these equal!

On the other hand, since $\vec R = \vec r\circ\vec G$

$$\iint_R (g\circ\vec R)(s,t)\,\|\vec R_s\times\vec R_t\|\,ds\,dt = \iint_R (g\circ\vec r)(\vec G(s,t))\left\|\frac{\partial}{\partial s}(\vec r\circ\vec G)\times\frac{\partial}{\partial t}(\vec r\circ\vec G)\right\|ds\,dt.$$

But $|\det J_{\vec G}| = \left|\det\begin{pmatrix}u_s & u_t\\ v_s & v_t\end{pmatrix}\right| = |u_sv_t - u_tv_s|$, while

$$\left\|\frac{\partial}{\partial s}(\vec r\circ\vec G)\times\frac{\partial}{\partial t}(\vec r\circ\vec G)\right\| = \|(u_s\vec r_u + v_s\vec r_v)\times(u_t\vec r_u + v_t\vec r_v)\|$$

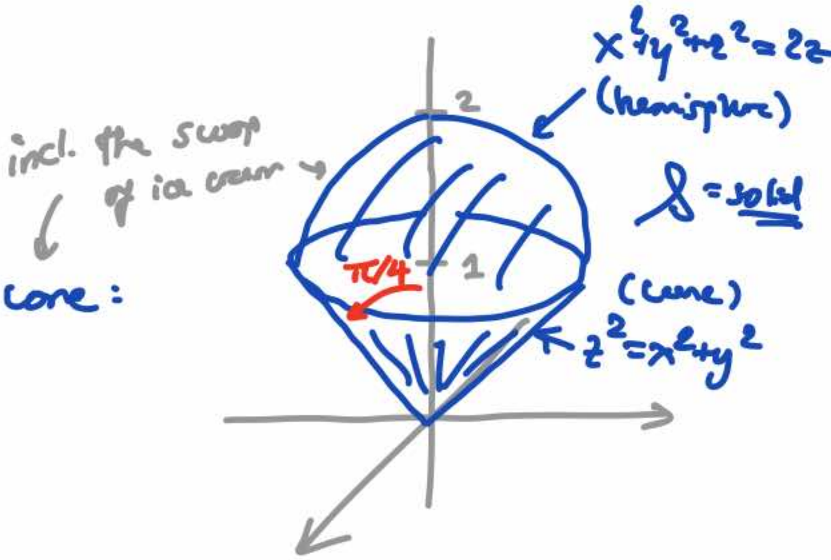
$$= \|u_su_t\underbrace{\vec r_u\times\vec r_u}_{0} + u_sv_t\,\vec r_u\times\vec r_v + v_su_t\underbrace{\vec r_v\times\vec r_u}_{-\vec r_u\times\vec r_v} + v_sv_t\underbrace{\vec r_v\times\vec r_v}_{0}\|$$

$$= |\det J_{\vec G}|\,\|\vec r_u\times\vec r_v\|.$$

$\square$

Here is one more good example of how to use spherical coordinates to compute triple integrals:

Ex/ Find the centroid of the solid ice cream cone:



First, we need its volume:

$$\iiint_{\mathcal{S}} 1\,dV = V(\mathcal{S}).$$

Note: hemisphere eqn. becomes $\rho^2 = 2\rho\cos\phi$

$\rho = 2\cos\phi$

Then the centroid's z-coordinate is $\bar{z} = \dfrac{\iiint_S z\,dV}{v(S)}$.

The other 2 coords. — $\bar{x}$ & $\bar{y}$ — are 0.

$$\iiint_S 1\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{2\cos\phi} \rho^2 \sin\phi \,d\rho\,d\theta\,d\phi$$

$$= 2\pi \int_0^{\pi/4} \sin\phi \underbrace{\left(\int_0^{2\cos\phi} \rho^2\,d\rho\right)}_{= \rho^3/3 \big|_0^{2\cos\phi} = \frac{8}{3}\cos^3\phi} d\phi$$

$$= \frac{16\pi}{3}\int_0^{\pi/4} \sin\phi\cos^3\phi\,d\phi = \frac{\cancel{16\pi}^{\,4}}{3}\left[-\frac{1}{\cancel{4}}\cos^4\phi\right]_0^{\pi/4}$$

$$= \frac{4\pi}{3}\left(1-\left(\frac{\sqrt{2}}{2}\right)^4\right) = \frac{4\pi}{3}\cdot\frac{3}{4} = \pi.$$

$$\iiint_S z\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{2\cos\phi} \rho^3 \cos\phi \sin\phi \,d\rho\,d\theta\,d\phi$$

$z = \rho\cos\phi$

$$= 2\pi\int_0^{\pi/4} \cos\phi\sin\phi \underbrace{\left(\int_0^{2\cos\phi}\rho^3\,d\rho\right)}_{\rho^4/4 \big|_0^{2\cos\phi} = 4\cos^4\phi} d\phi$$

$$= 8\pi\int_0^{\pi/4} \sin\phi\cos^5\phi\,d\phi$$

$$= 8\pi\left[-\frac{1}{6}\cos^6\phi\right]_0^{\pi/4} = \frac{8\pi}{6}\left(1-\left(\frac{\sqrt{2}}{2}\right)^6\right)$$

$$= \frac{\cancel{8}\pi}{6}\cdot\frac{7}{\cancel{8}} = \frac{7\pi}{6}$$

$$\Rightarrow \bar{z} = \frac{7\pi/6}{\pi} = \frac{7}{6}.$$

Of course, the entire thing has probably been devoured while we were figuring out how to balance it. //