Lecture 46 : Stokes's Theorem

Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field on $\mathbb{R}^3$ (or an open subset thereof). We define a new vector field by

$$\text{curl}(\vec{F}) := (R_y - Q_z,\ P_z - R_x,\ Q_x - P_y).$$

How to remember this? By abusing notation and pretending the gradient operator is a vector

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),$$

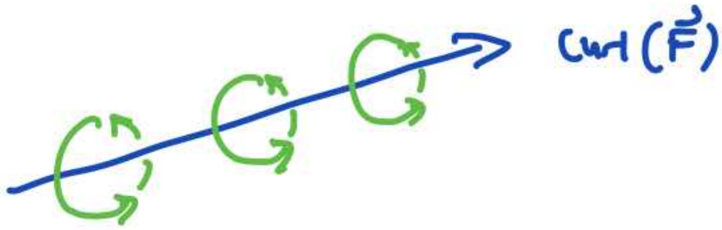
we write

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{e}_1 + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{e}_2 + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{e}_3$$

$$= \text{curl}(\vec{F}).$$

If we think of $\vec{F}$ as the velocity field of a fluid, then curl $(\vec{F})$ gives something like an axis about which the fluid rotates / "curls" most rapidly at each point, oriented according to the right-hand rule:

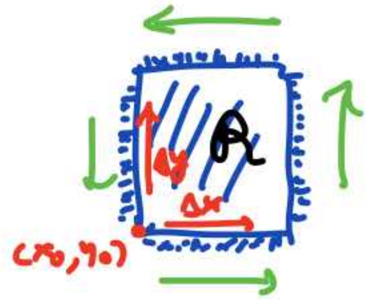


We say that $\vec{F}$ is irrotational if $\mathrm{curl}(\vec{F}) = \vec{0}$.

- Notice that if the domain of $\vec{F}$ is convex (or more generally, simply connected)

$$\vec{F} \text{ irrotational} \iff R_y = Q_z,\ R_x = P_z,\ Q_x = P_y$$
$$\overset{\text{Lect. 37}}{\iff} \vec{F} \text{ conservative}.$$

- For $\vec{F} = (P, Q)$ a velocity field in the plane, suppose we wanted to calculate the amount of counterclockwise rotation the rectangle shown would be subjected to. (independent of the overall direction of fluid flow)



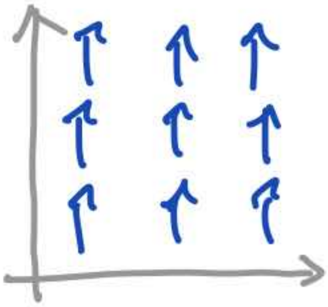
You need to consider

$$\big(Q(x_0+\Delta x, y_0) - Q(x_0, y_0)\big)\underbrace{\Delta y}_{\text{length of RHS/LHS}} - \big(P(x_0, y_0+\Delta y) - P(x_0, y_0)\big)\underbrace{\Delta x}_{\text{length of top/bottom}}$$

$$= (Q_x \Delta x)\Delta y - (P_y \Delta y)\Delta x = (Q_x - P_y)\, a(R).$$

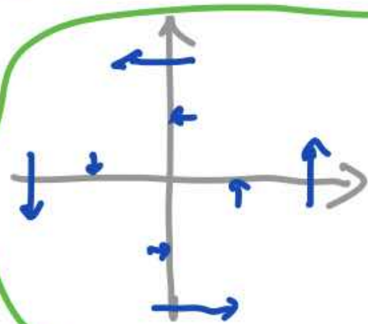
If we write $\vec{F} = (P, Q, 0)$ then $\mathrm{curl}(\vec{F}) = (Q_x - P_y)\hat{e}_3$.

Examples with $\vec{F} = (P, Q)$



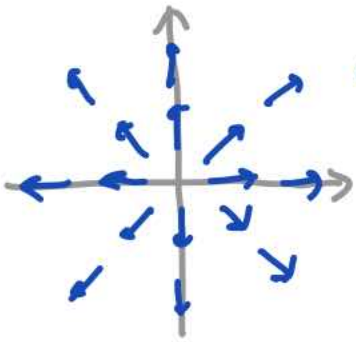
$\vec{F} = (0, c)$

$\text{curl}\, \vec{F} = \vec{0}$



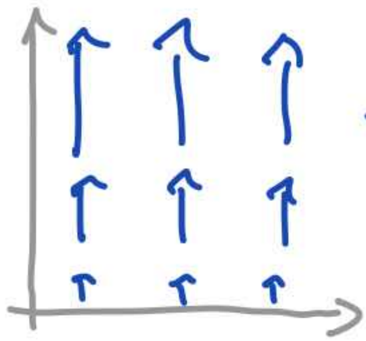
$\vec{F} = (-y, x)$

$\text{curl}\, \vec{F} = 2\hat{e}_3$



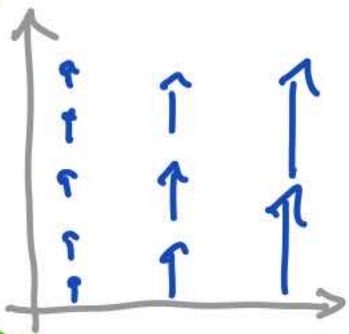
$\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$\text{curl}\, \vec{F} = \vec{0}$



$\vec{F} = (0, y)$

$\text{curl}\, \vec{F} = \vec{0}$



$\vec{F} = (0, x)$

$\text{curl}\, \vec{F} = \hat{e}_3$

not irrotational

- One more fun remark here: for a C^1 function f,

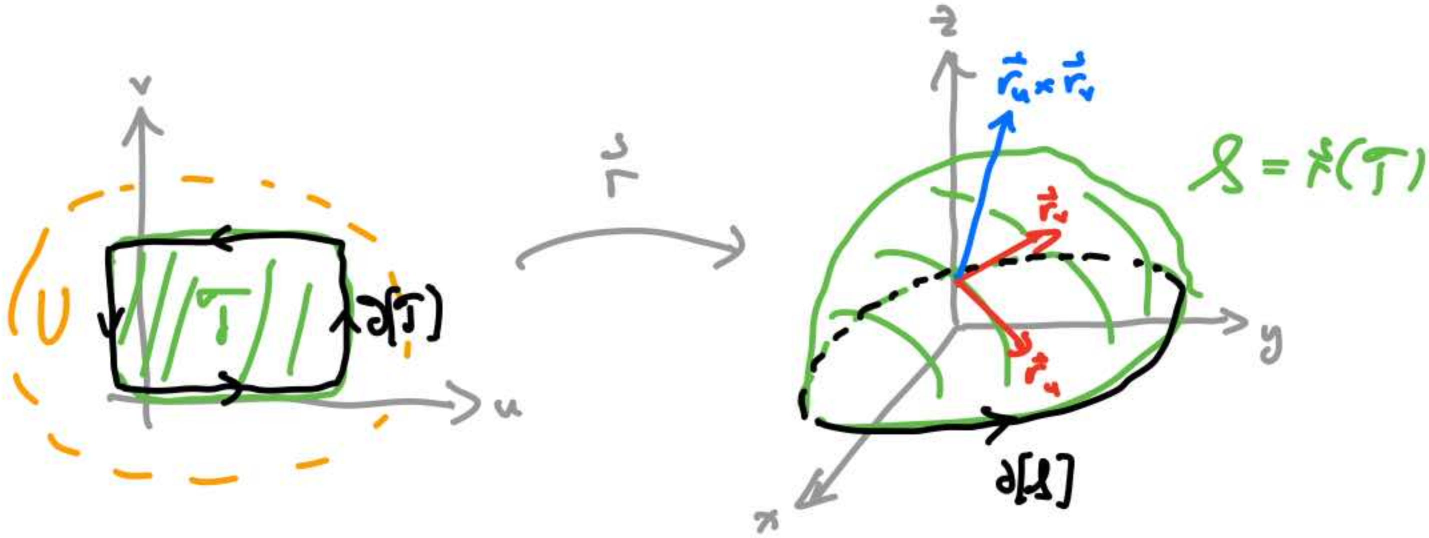
$$\text{curl}(\vec{\nabla} f) = \vec{\nabla} \times \vec{\nabla} f = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right)$$

$$= \vec{0} \quad \text{(Clairaut)}$$

Which reproves "$\vec{F}$ conservative (i.e. $= \vec{\nabla} f$) $\Rightarrow$ curl $\vec{F} = \vec{0}$" above.

We return to the scenario of yesterday's lecture:



with $\vec r : U \to \mathbb{R}^3$ now assumed C^2. In particular, $\vec r$ maps $\mathcal{T}$ to $\mathcal{S}$ and $\partial[\mathcal{T}]$ to $\partial[\mathcal{S}] = \mathcal{C}$ in 1-to-1 fashion. Let $\vec F = (P, Q, R)$ be a C^1 vector field on $\mathcal{S}$ (or a set containing it).

Theorem (Stokes):

$$\iint_{\mathcal{S}} \overbrace{\mathrm{Curl}(\vec F)}^{(R_y - Q_z,\ P_z - R_x,\ Q_x - P_y)} \cdot \hat n \, dS = \oint_{\partial[\mathcal{S}]} \overbrace{\vec F}^{(P, Q, R)} \cdot d\vec r .$$

$$\iint_{\mathcal{S}} \left[(R_y - Q_z)\, dy \wedge dz + (P_z - R_x)\, dz \wedge dx + (Q_x - P_y)\, dx \wedge dy \right] \qquad = \qquad \int_{\mathcal{C}} P\,dx + Q\,dy + R\,dz$$

Claim:

$$\iint_{\mathcal{S}} \left(P_z\, dz \wedge dx - P_y\, dx \wedge dy \right) \overset{①}{=} \int_{\mathcal{C}} P\,dx$$

$$+$$

$$\iint_{\mathcal{S}} \left(Q_x\, dx \wedge dy - Q_z\, dy \wedge dz \right) \overset{②}{=} \int_{\mathcal{C}} Q\,dy$$

$$+$$

$$\iint_{\mathcal{S}} \left(R_y\, dy \wedge dz - R_x\, dz \wedge dx \right) \overset{③}{=} \int_{\mathcal{C}} R\,dz$$

Proof: We will check ①, i.e.

$$(*) \quad \iint_S P_z\, dz\wedge dx - P_y\, dx\wedge dy = \int_C P\, dx$$

$$\text{LHS}(*) = \iint_T \left(P_z(\vec{r}(u,v)) \frac{\partial(z,x)}{\partial(u,v)} - P_y(\vec{r}(u,v)) \frac{\partial(x,y)}{\partial(u,v)} \right) du\wedge dv$$

$$= \iint_T \left(P_z z_u x_v - P_z z_v x_u - P_y x_u y_v + P_y x_v y_u \right) du\, dv$$

cancel

$$= \iint_T \left\{ \underbrace{(P_x x_u + P_y y_u + P_z z_u)}\, x_v - \underbrace{(P_x x_v + P_y y_v + P_z z_v)}\, x_u \right\} du\, dv$$

$$= \iint_T \left\{ x_v \frac{\partial}{\partial u}(P\circ\vec{r}) - x_u \frac{\partial}{\partial v}(P\circ\vec{r}) \right\} du\, dv$$

change of variable

$$\text{RHS}(*) = \int_C P\, dx = \int_{\partial[T]} (P\circ\vec{r})(x_u\, du + x_v\, dv)$$

$$= \int_{\partial[T]} [(P\circ\vec{r})\, x_u]\, du + [(P\circ\vec{r})\, x_v]\, dv$$

Green's Thm.

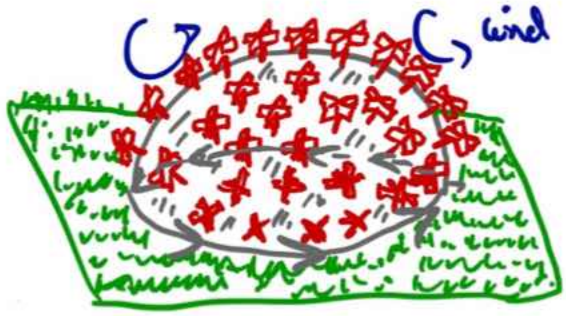
$$= \iint_T \left\{ \frac{\partial}{\partial u}[(P\circ\vec{r})\, x_v] - \frac{\partial}{\partial v}[(P\circ\vec{r})\, x_u] \right\} du\, dv$$

$$= \iint_T \left\{ x_v \frac{\partial}{\partial u}(P\circ\vec{r}) - x_u \frac{\partial}{\partial v}(P\circ\vec{r}) + \underbrace{(P\circ\vec{r})\, x_{vu} - (P\circ\vec{r})\, x_{uv}}_{=0 \text{ (by Clairaut)}} \right\} du\, dv$$

$$= \iint_T \left\{ x_v \frac{\partial}{\partial u}(P\circ\vec{r}) - x_u \frac{\partial}{\partial v}(P\circ\vec{r}) \right\} du\, dv .$$

$\square$

Ex 1/ What if people could design a roof that generated an electrical current proportional to the integral $\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS$, where $\vec{F}$ is the wind-velocity field ... and wanted to sit it on the ground (where there is no wind)?

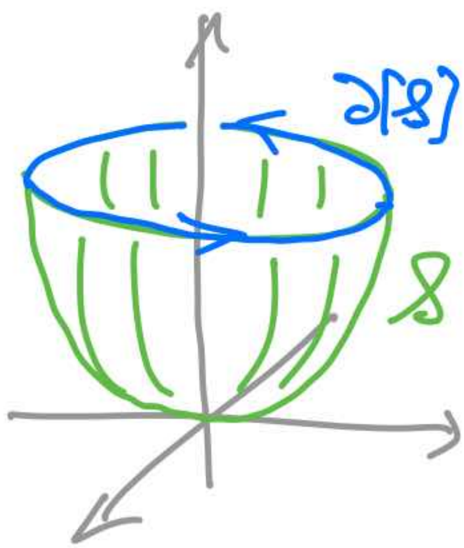


Well, better luck with the next idea, because

$$\vec{F}\equiv\vec{0} \text{ on } \partial S \implies \oint_{\partial[S]}\vec{F}\cdot d\vec{r}=0 \underset{\text{Stokes}}{\implies} \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,dS=0.$$ //

Conversely, Stokes tells us yet again that if $\operatorname{curl}(\vec{F})=\vec{0}$, then $\oint_C \vec{F}\cdot d\vec{r}=0$ on closed paths.

Ex 2/ Let's verify that Stokes works in one computational example:



$\vec{F}=(y,-x,yz)$

$S=\{(x,y,z)\mid z\le 1,\ z=x^2+y^2\}$

$\partial S=\{(x,y,z)\mid x^2+y^2=1,\ z=1\}$

$\underline{\underline{\text{RHS}}}$: $\oint_{\partial[S]}\vec{F}\cdot d\vec{r}=\oint_{\partial[S]} y\,dx - x\,dy + yz\,\cancel{dz}^{\,0}$

$\vec{r}(t)=(\cos t,\sin t,1)$ $=\int_0^{2\pi}\{\sin t(-\sin t\,dt)-\cos t(\cos t\,dt)\}$

$$= -\int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = -2\pi.$$

<u>LHS</u> :

$$\operatorname{curl} \vec{F} = (z, 0, -2)$$

$$\vec{r}(u,v) = (u, v, u^2+v^2) \quad \text{where } T = \{u^2+v^2 \leq 1\}$$

$$\vec{r}_u = (1, 0, 2u), \quad \vec{r}_v = (0, 1, 2v)$$

$$\vec{r}_u \times \vec{r}_v = (-2u, -2v, 1)$$

$$\Longrightarrow (\operatorname{curl} \vec{F})(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = (u^2+v^2, 0, -2) \cdot (-2u, -2v, 1)$$

$$= -2u(u^2+v^2) - 2$$

$$\Longrightarrow \iint_S \operatorname{curl}(\vec{F}) \cdot \hat{n}\, dS = \iint_T \Big(-2 \underset{r\cos\theta}{\underset{\uparrow}{u}} \big(\underbrace{u^2+v^2}_{r^2}\big) - 2\Big) \underbrace{du\, dv}_{r\, dr\, d\theta}$$

$$= -2 \int_0^{2\pi} \int_0^1 (r^4 \cos\theta + r)\, dr\, d\theta$$

$$= -2 \int_0^{2\pi} \Big(\frac{1}{5} \underbrace{\cos\theta}_{\text{integrates to } 0} + \frac{1}{2}\Big)\, d\theta$$

$$= -2 \cdot \frac{1}{2} \cdot 2\pi = -2\pi.$$

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