Lecture 47: Div, grad, & curl

Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field on $\mathbb{R}^3$ (or an open subset thereof). We define a new FUNCTION by

$$\mathrm{div}(\vec{F}) := P_x + Q_y + R_z,$$

the divergence of $\vec{F}$ (rather easier to remember than the curl!).

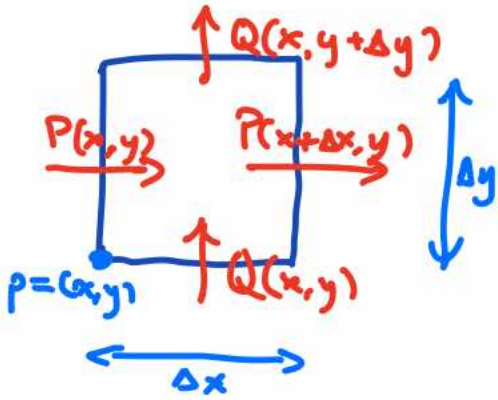
Writing as before $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$, by abuse of notation

$$\begin{aligned} \vec{\nabla}\cdot\vec{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(P,Q,R) \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \mathrm{div}(\vec{F}). \end{aligned}$$

A vector field with zero divergence is called solenoidal or incompressible. The intuition for the latter term (in the context of fluid flow) is as follows, working for simplicity with $\vec{F} = (P,Q)$ in the plane:

the total flux out of a little rectangle is

$$\approx \Delta y\,\big(P(x+\Delta x, y) - P(x,y)\big) + \Delta x\,\big(Q(x,y+\Delta y) - Q(x,y)\big).$$



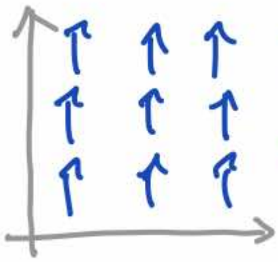
Dividing by $\Delta A = (\Delta x)(\Delta y)$ and taking the limit as $\Delta x, \Delta y \to 0$, we get (for flux per unit area at p)

$$\lim_{\Delta x \to 0} \frac{P(x+\Delta x, y) - P(x,y)}{\Delta x} + \lim_{\Delta y \to 0} \frac{Q(x,y+\Delta y) - Q(x,y)}{\Delta y}$$

$$= \frac{\partial P}{\partial x}(p) + \frac{\partial Q}{\partial y}(p) = \operatorname{div}(\vec{F})(p). \qquad (*)$$

So if $(*) > 0$ [resp. $(*) < 0$] the fluid tends to diverge from [resp. accumulate at] p; and $(*) = 0$ everywhere would mean that the fluid density cannot change (inflow = outflow), hence "incompressible".

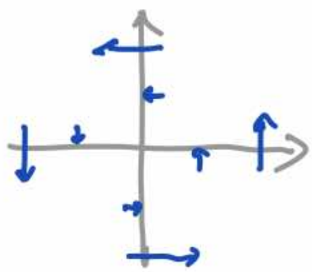
Examples from Lecture 46



$\vec{F} = (0,c)$

$\operatorname{curl} \vec{F} = \vec{0}$ irrotational

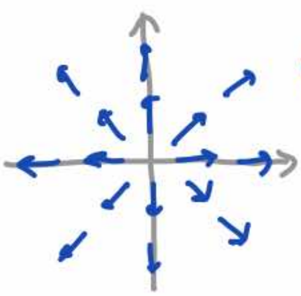
$\operatorname{div} = 0$ incompressible



$\vec{F} = (-y, x)$

$\operatorname{curl} \vec{F} = 2\hat{e}_3$

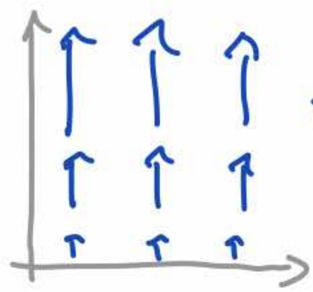
$\operatorname{div} = 0$ incompressible



$\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$

$\operatorname{curl} \vec{F} = \vec{0}$ irrotational

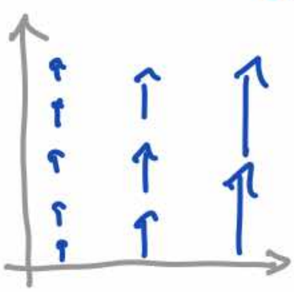
$\operatorname{div} = \frac{1}{\sqrt{x^2+y^2}}$



$\vec{F} = (0, y)$

$\operatorname{curl} \vec{F} = \vec{0}$ irrotational

$\operatorname{div} = 1$



$\vec{F} = (0, x)$

$\operatorname{curl} \vec{F} = \hat{e}_3$

$\operatorname{div} = 0$ incompressible

Last time we showed that

(1) $\operatorname{curl}(\vec{\nabla} f) = \vec{0}$, i.e. CONSERVATIVE FIELDS ARE IRROTATIONAL

Another key identity is

(2) $\underbrace{\operatorname{div}(\operatorname{curl}\vec{F})}$ $= 0$:

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

the 2 "equal" rows already suggest 0 (though these aren't really vectors, so not yet done)

$$= \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y)$$

$$= \cancel{R_{yx}} - \cancel{Q_{zx}} + \cancel{P_{zy}} - \cancel{R_{xy}} + \cancel{Q_{xz}} - \cancel{P_{yz}}$$

$$= 0.$$

Recall that (1) had a converse: writing $\mathcal{D}$ for the domain of $\vec{F}$,

Theorem A: If $\mathcal{D}$ is $\underbrace{\text{simply connected}}$, then $\vec{F}$ is conservative $\Leftrightarrow$ $\vec{F}$ is irrotational.

"in 3-space, this means "no tunnels through $\mathcal{D}$""

(2) has one too:

Theorem B: If $\mathcal{D}$ is $\overbrace{\text{2-connected}}^{\text{no "solid holes"}}$, then $\vec{F}$ is the curl of some other vector field $\Leftrightarrow$ $\vec{F}$ is incompressible.

Proof for $\mathcal{D} = [\vec{a}, \vec{b}]$ (rectangular box): (2) gives "$\Rightarrow$".

We must show "$\Leftarrow$": given $\vec{F} = (P, Q, R)$ with $P_x + Q_y + R_z = 0$, construct $\vec{G} = (A, B, C)$ s.t. $C_y - B_z = P$, $A_z - C_x = Q$, $B_x - A_y = R$.

In fact, we can do this with $\vec{G} = (0, B, C)$: taking

$C(x,y,z) := -\int_{x_0}^{x} Q(t,y,z)\,dt$ & $B(x,y,z) := \int_{x_0}^{x} R(t,y,z)\,dt + g(y,z)$,

we have $-C_x = Q$ & $B_x = R$, and

$$C_y - B_z = -\int_{x_0}^{x} Q_y(t,y,z)\,dt - \int_{x_0}^{x} R_z(t,y,z)\,dt - g_z(y,z)$$

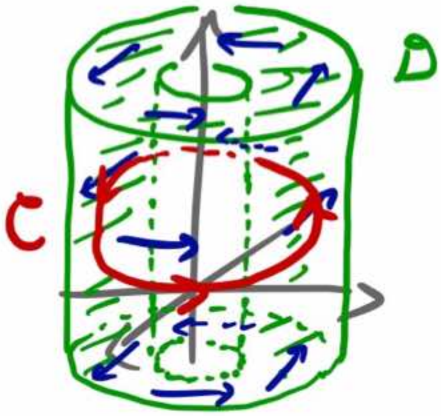
use $P_x + Q_y + R_z = 0$

$$= \int_{x_0}^{x} P_x(t,y,z)\,dt - g_z(y,z)$$

$$= P(x,y,z) - P(x_0,y,z) - g_z(y,z).$$

So taking $g(y,z) = -\int_{z_0}^{z} P(x_0,y,t)\,dt$ solves $C_y - B_z = P$. $\square$

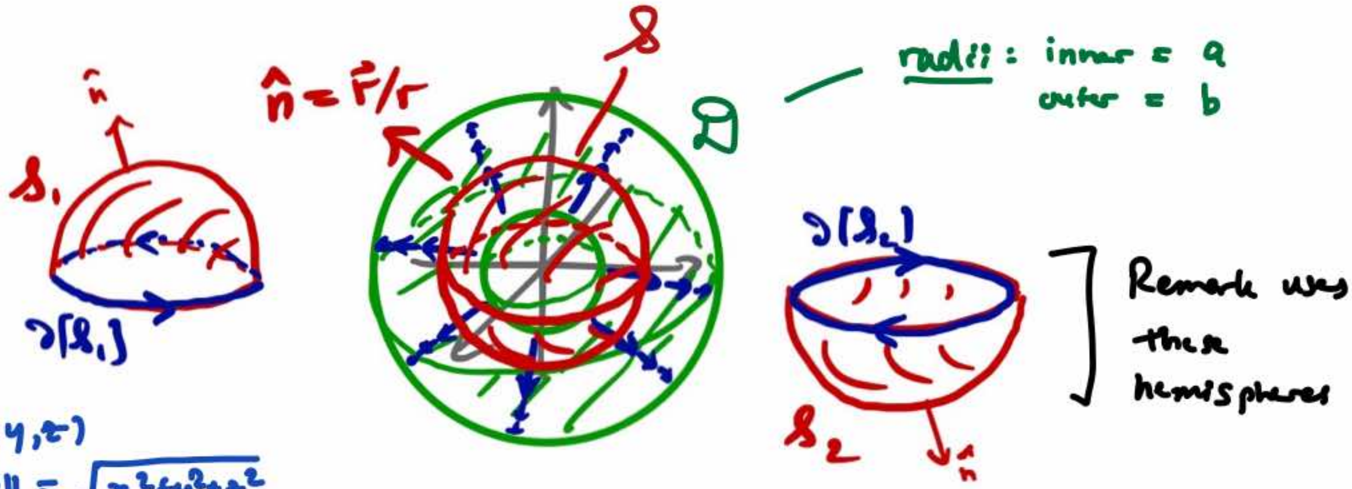
Next recall that if $\mathcal{D}$ is not simply connected, then "$\Longleftarrow$" in Thm. A <u>fails</u>:



$\vec{F} = \left(\frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0 \right)$ is actually irrotational ($\operatorname{curl} \vec{F} = \vec{0}$),

but $\oint_{\mathcal{C}} \vec{F} \cdot d\vec{r} = 2\pi \neq 0$ means that $\vec{F}$ is <u>not</u> conservative/a gradient field.

In Theorem B, if $\mathcal{D}$ is not 2-connected, then "$\Longleftarrow$" fails:



$$\begin{cases} \vec{r} = (x,y,z) \\ r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2} \end{cases}$$

In the picture, $\mathcal{D}$ is a hollow ball and $\vec{F} = \left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right) = r^{-3}\vec{r}$ is incompressible:

$$\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(r^{-3}\vec{r}) = (\vec{\nabla} r^{-3})\cdot\vec{r} + r^{-3}(\vec{\nabla}\cdot\vec{r})$$
$$= -3r^{-5}\vec{r}\cdot\vec{r} + r^{-3}3 = -3r^{-3} + 3r^{-3} = 0.$$

Suppose $\vec{F} = \text{curl}\,\vec{G}$. Then on a sphere $\mathcal{S}$ of radius $R \in (a,b)$,

$$\iint_{\mathcal{S}} \vec{F}\cdot\hat{n}\,dS \overset{\text{Stokes}}{=\!=} \underbrace{\int_{\partial[\mathcal{S}]} \vec{G}\cdot d\vec{r}}_{=0 \text{ b/c sphere has no boundary!}} = 0$$

$$\Vert$$

$$\iint_{\mathcal{S}} \frac{\vec{r}}{r^3}\cdot\frac{\vec{r}}{r}\,dS = \iint_{\mathcal{S}} \frac{r^2}{r^4}\,dS = \frac{1}{R^2}\iint_{\mathcal{S}} dS = \frac{1}{R^2}\overbrace{a(\mathcal{S})}^{4\pi R^2} = 4\pi,$$

a contradiction.

Remark: We should be careful here; since $\mathcal{S}$ doesn't have a 1-to-1 parametrization, break it into hemispheres as shown & use Stokes's Theorem on both halves. The end result is the same b/c $\partial[\mathcal{S}_1] = -\partial[\mathcal{S}_2]$. One only gets in trouble with non-orientable surfaces; see Apostol §12.18 for a brief discussion.

Having discussed $\operatorname{curl}(\vec{\nabla} f)=\vec{0}$ and $\operatorname{div}(\operatorname{curl}\vec{F})=0$, what about

- $\operatorname{curl}(\operatorname{curl}\vec{F})$
- $\operatorname{div}(\vec{\nabla} f)$

?

Note that "$\operatorname{curl}(\operatorname{div}\vec{F})$" and "$\vec{\nabla}(\operatorname{curl}\vec{F})$" make no sense; "$\vec{\nabla}(\operatorname{div}\vec{F})$" will come up below briefly.

You may recall the identity

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

for vectors in $\mathbb{R}^3$, which can also be written as $\vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}$. Together with our abuse of notation, this suggests

(3)
$$\underbrace{\vec{\nabla}\times(\vec{\nabla}\times\vec{F})}_{\text{i.e. } \operatorname{curl}(\operatorname{curl}\vec{F})} = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{F}$$
$$= \vec{\nabla}(\operatorname{div}\vec{F}) - \nabla^2\vec{F}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacian. Let's check this:

If $\vec{F}=(P,Q,R)$, then $\vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}\times(R_y-Q_z,\, P_z-R_x,\, Q_x-P_y)$

$$=\big((Q_x-P_y)_y-(P_z-R_x)_z,\ (R_y-Q_z)_z-(Q_x-P_y)_x,\ (P_z-R_x)_x-(R_y-Q_z)_y\big)$$

$$=\big(Q_{xy}+R_{xz}+P_{xx}-P_{xx}-P_{yy}-P_{zz},\ R_{yz}+P_{yx}+Q_{yy}-Q_{xx}-Q_{yy}-Q_{zz},$$
$$P_{zx}+Q_{zy}+R_{zz}-R_{xx}-R_{yy}-R_{zz}\big)$$

$$=\big((P_x+Q_y+R_z)_x,\ (P_x+Q_y+R_z)_y,\ (P_x+Q_y+R_z)_z\big)$$
$$-\big(P_{xx}+P_{yy}+P_{zz},\ Q_{xx}+Q_{yy}+Q_{zz},\ R_{xx}+R_{yy}+R_{zz}\big)$$

$$=\vec{\nabla}(P_x+Q_y+R_z)-\nabla^2(P,Q,R). \qquad \square$$

Finally,

(4) $\operatorname{div}(\vec{\nabla} f) = \operatorname{div}(f_x, f_y, f_z) = f_{xx} + f_{yy} + f_{zz}$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f = \nabla^2 f$$

is the Laplacian of f. When $\nabla^2 f = 0$, f is called a harmonic function. The same notion exists for vector fields.

AN APPLICATION TO ELECTROMAGNETISM

Maxwell's equations (for a vacuum) describe the relationship between the electric & magnetic fields $\vec{E}$ and $\vec{B}$ (which depend on x, y, z and t):

- $\operatorname{div} \vec{E} = 0 = \operatorname{div} \vec{B}$
- $\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, $\operatorname{curl} \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

These explain, for example, why a current in a coil

(a) can move a magnet [solenoid]

(b) encounters "magnetic inertia" opposing any change to the electric current [inductor].

Notice that

$$\nabla^2 \vec{E} = \vec{\nabla}\big(\cancel{\operatorname{div} \vec{E}}^{\,0}\big) - \operatorname{curl}\big(\underbrace{\operatorname{curl} \vec{E}}_{-\frac{\partial \vec{B}}{\partial t}}\big) = \operatorname{curl}\left(\frac{\partial \vec{B}}{\partial t}\right) = \frac{\partial}{\partial t} \operatorname{curl} \vec{B}$$

$$= \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \quad \text{\& similarly } \nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}.$$

These are exactly the PDEs for a wave propagating in 3-space with speed $\frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. Since the experimentally observed values of μ_0, ε_0, and c = speed of light were such that $\frac{1}{\sqrt{\mu_0 \varepsilon_0}} \sim c$, Maxwell hypothesized that light is an electromagnetic wave.

We also see from $\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$ that if $\vec{E}$ is not changing in time, it is harmonic. In electrostatics this allows you to solve for an electric field with given boundary conditions.