Lecture 48: Gauss's Theorem

Yesterday we estimated that the flux of a vector field in $\mathbb{R}^2$ through the boundary of a small rectangle, as the area of the rectangle times the local value of the divergence. This becomes exact as the rectangle shrinks to a point. Here's a version of this same observation, but in 3-space, which we'd like to prove carefully:

<u>Proposition</u>: Let $\vec{F} = (P, Q, R)$ be C^1 on a connected open set $U \subset \mathbb{R}^3$, & B_r denote the ball of radius r about $\vec{a} \in U$. Then

$$(\operatorname{div} \vec{F})(\vec{a}) = \lim_{r \to 0} \frac{1}{v(B_r)} \iint_{\partial[B_r]} \vec{F} \cdot \hat{n} \, dS .$$

Sphere of radius r

<u>Attempt at Proof</u>: Since $\vec{F}$ is C^1,

$$g := \operatorname{div} \vec{F} = P_x + Q_y + R_z$$

is continuous. So given $\epsilon > 0$, $\exists \delta > 0$ s.t.

$$\|\vec{x}-\vec{a}\| < \delta \implies |g(\vec{x})-g(\vec{a})| < \frac{\epsilon}{2}.$$

In particular, if $r < \delta$ this holds for all $\vec{x} \in \overline{B_r}$ (closed ball). Therefore

$$\begin{aligned}
\epsilon > \frac{\epsilon}{2} &\geq \frac{1}{v(B_r)} \iiint_{B_r} |g(\vec{x})-g(\vec{a})|\, dV \\
&\geq \frac{1}{v(B_r)} \left| \iiint_{B_r} (g(\vec{x})-g(\vec{a}))\, dV \right| \\
&= \frac{1}{v(B_r)} \left| \iiint_{B_r} \operatorname{div}\vec{F}\, dV - g(\vec{a})\, v(B_r) \right| \\
&= \left| \underbrace{\frac{1}{v(B_r)} \iiint_{B_r} \operatorname{div}\vec{F}\, dV}_{} - (\operatorname{div}\vec{F})(\vec{a}) \right|
\end{aligned}$$

To finish the proof, we need this to equal the "flux"

$$\iint_{\partial[B_r]} \vec{F}\cdot\hat{n}\, dS. \qquad \boxed{?}$$

That is, we need Gauss's Theorem.

Before stating it, we need to revisit surface integrals a bit.

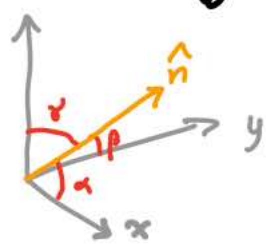
Let $\mathcal{S} \subset \underset{x,y,z}{\mathbb{R}^3}$ be a surface smoothly parametrized by $\vec{r}: \mathcal{T} \to \mathbb{R}^3$, where $\mathcal{T} \subset \underset{u,v}{\mathbb{R}^2}$. Recall that

- $$\iint_{\mathcal{S}} f(x,y,z)\, dS := \iint_{\mathcal{T}} f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$$

- $$\iint_S f(x,y,z)\, \underset{\substack{dy\wedge dz \\ \vdots}}{dx\wedge dy} := \iint_{\sigma} f(\vec{r}(u,v))\, \underset{\frac{\partial(y,z)}{\partial(u,v)}}{\frac{\partial(x,y)}{\partial(u,v)}}\, du\, dv$$

- $$\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_{\sigma} \vec{F}(\vec{r}(u,v)) \cdot \frac{(\vec{r}_u\times\vec{r}_v)}{\cancel{\|\vec{r}_u\times\vec{r}_v\|}}\cancel{\|\vec{r}_u\times\vec{r}_v\|}\, du\, dv$$
$$= \iint_{\sigma} \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\, dv$$
$$= \iint_S P\, dy\wedge dz + \iint_S Q\, dz\wedge dx + \iint R\, dx\wedge dy .$$

Now write $\hat{n} = (\cos\alpha, \cos\beta, \cos\gamma)$
and observe that



$$\iint_S P\, dy\wedge dz = \iint_{\sigma} P \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} du\, dv = \iint_{\sigma} \begin{vmatrix} P & 0 & 0 \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} du\, dv$$
$$= \iint_S (P,0,0)\cdot \underbrace{\hat{n}\, dS}_{(\cos\alpha,\,\cos\beta,\,\cos\gamma)} = \iint_S P \cos\alpha\, dS .$$

WARNING: This varies as you move over S!

Similarly,
$$\iint_S Q\, dz\wedge dx = \iint_S Q\cos\beta\, dS \quad \text{and} \quad \iint_S R\, dx\wedge dy = \iint_S R\cos\gamma\, dS .$$

GAUSS'S THEOREM: Let $\vec{F} = (P,Q,R)$ be a C^1 vector field on $D \subset \mathbb{R}^3$, and $V \subset D$ be a (solid) region with smooth boundary (surface) $S = \partial[V]$. Then
$$\iiint_V \operatorname{div}\vec{F}\, dV = \iint_S \vec{F}\cdot\hat{n}\, dS .$$

Ex 1/ Compute the flux of $\vec{F} = (x, y, z)$ through the unit sphere $S : x^2+y^2+z^2 = 1$.

We actually did this in Lecture 45, but Gauss makes it easier: $\operatorname{div}\vec{F} = P_x + Q_y + R_z = 1+1+1 = 3 \underset{\text{Gauss}}{\Longrightarrow}$

$$\text{flux} = \iiint_{\binom{\text{unit}}{\text{ball}}} 3\,dV = 3v\binom{\text{unit}}{\text{ball}} = 3\cdot\frac{4}{3}\pi = 4\pi.$$

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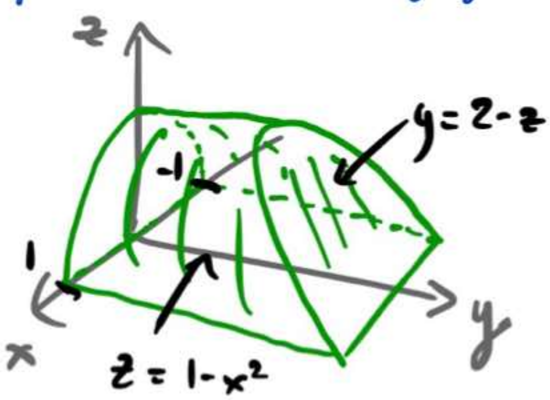
Ex 2/ Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\vec{F} = (xy, y^2+e^{xz^2}, \sin(xy))$

and S is the surface bounded by $z=0$, $y=0$, $z=1-x^2$, and $y+z=2$.



- $\operatorname{div}\vec{F} = (xy)_x + (y^2+e^{xz^2})_y + (\sin xy)_z = y + 2y + 0 = 3y$
- $\iint_S \vec{F}\cdot\hat{n}\,dS \underset{\text{Gauss}}{=} \iiint_V 3y\,dV = \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y\,dy\,dz\,dx$

$$= \int_{-1}^{1}\int_0^{1-x^2} \frac{3}{2}(2-z)^2\,dz\,dx = \int_{-1}^{1}\Big[-\frac{1}{2}(2-z)^3\Big]_{z=0}^{z=1-x^2} dx$$

$$= \int_{-1}^{1}\Big(4-\frac{1}{2}(1+x^2)^3\Big)\,dx = \cdots = \frac{184}{25}.$$

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Ex 3/ Define $\vec{F} = (0,0,cz)$ on $\mathbb{R}^3$, and let

$$\mathcal{V} \subset \{ (x,y,z) \mid z \le 0 \}.$$

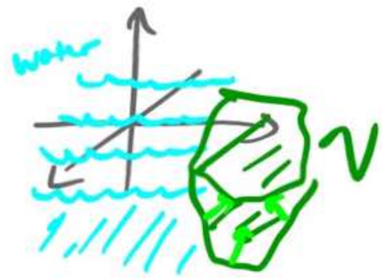
Think of $\vec{F}$ as the downward pressure of a fluid of density c.

Since a fluid exerts equal pressures in all directions, we define the buoyant force on $\mathcal{V}$, due to the fluid, as

$$-\iint_{\partial[\mathcal{V}]} \vec{F} \cdot \hat{n} \, dS \quad \left(= \iint_{\partial[\mathcal{V}]} \hat{e}_3 \cdot \underbrace{cz\hat{n}}_{\substack{\text{pressure}\\ \text{on surface}\\ \text{of } \mathcal{V}}} \, dS \right)$$

with inward-pointing normal

vertical component



Prove Archimedes's Theorem: the buoyant force on $\mathcal{V}$ is equal to the weight of the fluid displaced by $\mathcal{V}$.

First, to match the outward normal in Gauss's Theorem we need to reverse the sign of $\hat{n}$; so

$$\begin{aligned}
\text{buoyant force} &= \iint_{\partial[\mathcal{V}]} \vec{F} \cdot \hat{n} \, dS \\
&\underset{\text{Gauss}}{=} \iiint_{\mathcal{V}} \underbrace{\operatorname{div} \vec{F}}_{=c} \, dV \\
&= c \iiint_{\mathcal{V}} dV \\
&= c\, v(\mathcal{V}) \\
&= \text{weight of fluid displaced by } \mathcal{V}.
\end{aligned}$$

PROOF OF GAUSS: STEP 1 (Break in three)

$$
\begin{array}{ccc}
\text{LHS} & & \text{RHS} \\
\parallel & \text{Claim:} & \parallel \\
\iiint_V P_x\, dV & \overset{(1)}{=\!=\!=} & \iint_S P\, dy\wedge dz \\
+ & & + \\
\iiint_V Q_y\, dV & \overset{(2)}{=\!=\!=} & \iint_S Q\, dz\wedge dx \\
+ & & + \\
\iiint_V R_z\, dV & \overset{(3)}{=\!=\!=} & \iint_S R\, dx\wedge dy\ .
\end{array}
$$

STEP II (Prove (1)–(3) for V convex)

By "symmetry" it suffices to prove (3).

Since V is convex, it takes the form

$\{(x,y,z) \mid (x,y) \in T,\ g(x,y) \le z \le f(x,y)\}$,

and $\partial[V] = S_+ + S_0 + S_-$, as shown.

$$\iint_S R\, dx\wedge dy = \iint_S R\cos\gamma\, dS = \iint_{S_+} R\cos\gamma\, dS + \iint_{S_-} R\cos\gamma\, dS + \iint_{S_0} R\cancelto{0 \text{ on } S_0}{\cos\gamma}\, dS$$

$$= \iint_{S_+} R\, dx\wedge dy + \iint_{S_-} R\, dx\wedge dy$$

$$= \iint_T R(u,v,f(u,v))\, \cancelto{1}{\frac{\partial(x,y)}{\partial(u,v)}}\, du\, dv \ominus \iint_T R(u,v,g(x,y))\, \cancelto{1}{\frac{\partial(x,y)}{\partial(u,v)}}\, du\, dv$$

$\vec{r}(u,v) = (u,v,f(u,v))$ resp. $(u,v,g(u,v))$

b/c parametrization gives upward normal

$$= \iint_T \left(\int_{g(x,y)}^{f(x,y)} R_z(x,y,z)\, dz \right) dx\, dy = \iiint_V R_z\, dV.$$

$\boxed{\text{STEP III (Extend to arbitrary } \mathcal{V})}$

Chop $\mathcal{V}$ into convex sub-solids

$$\mathcal{V} = \sum \mathcal{V}_j \,,$$

then apply Gauss to each $\mathcal{V}_j$ and sum:

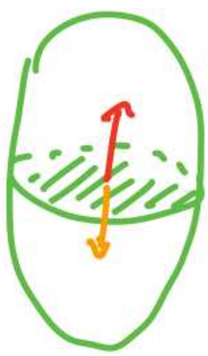
$$\sum_j \iiint_{\mathcal{V}_j} \operatorname{div} \vec{F} \, dV = \sum_j \iint_{\partial[\mathcal{V}_j]} \vec{F} \cdot \hat{n} \, dS \,.$$

$$\parallel$$

$$\iiint_{\mathcal{V}} \operatorname{div} \vec{F} \, dV$$

The main point now is that the surface integrals where one does the gluing have opposite normals, so that they cancel and the RHS indeed becomes

$$\iint_{\partial[\mathcal{V}]} \vec{F} \cdot \hat{n} \, dS \,.$$



$\square$