Today I want to tell you how a change in perspective unifies Gauss, Stokes, and the Fundamental Theorem of Calculus for line integrals. I've already suggested why they are all analogous; now we'll introduce the 'calcus' which establishes their unity and makes higher-dimensional generalizations immediate.

**Vector Spaces** *(real, of course)*

- $\mathbb{R}^0 := \{1\}$
- $\mathbb{R}^1 := 1\text{-dim' vector space with basis } \{dx\}$
- $\mathbb{R}^2 := 2\text{-dim' vector space with basis } \{dy, dx\}$
- $\mathbb{R}^3 := 3\text{-dim' vector space with basis } \{dy \wedge dx, dx \wedge dz, dz \wedge dy\}$
- $\mathbb{R}^4 := 4\text{-dim' vector space with basis } \{dz \wedge dy \wedge dx \wedge \text{etc.}\}$
- $\mathbb{R}^5$ & higher are $\mathbb{R}^0$.

**Differential forms** U $\subset \mathbb{R}^3$ open set

A (smooth) differential k-form is a $(C^\infty, \text{vector-valued})$ function from $U$ to $\mathbb{R}^k$. We write

$$\omega \in \Omega^k(U).$$

$k=0$: A 0-form is a $C^\infty$ function $f(x,y,z)$. That is, $\Omega^0(U) = C^\infty(U)$. 


$k=1$: A 1-form $\omega \in \mathcal{A}^1(U)$ takes the form

$$\omega = P(x,y,z) \, dx + Q(x,y,z) \, dy + R(x,y,z) \, dz.$$ 

$k=2$: A 2-form $\omega \in \mathcal{A}^2(U)$ takes the form

$$\omega = P(x,y,z) \, dy \wedge dz + Q(x,y,z) \, dx \wedge dx + R(x,y,z) \, dx \wedge dy.$$ 

$k=3$: A 3-form $\omega \in \mathcal{A}^3(U)$ takes the form

$$\omega = f(x,y,z) \, dx \wedge dy \wedge dz.$$ 

\[N.B. \] $k$ is called the degree of a differential $(k-1)$-form.

**Exterior derivative**

If $f \in C^0(U)$ is a function, define a 1-form $df \in \mathcal{A}^1(U)$ by

$$df := f_x \, dx + f_y \, dy + f_z \, dz.$$ 

We can extend this to higher degree, obtaining (for each $k$) linear maps

$$d : \mathcal{A}^k(U) \to \mathcal{A}^{k+1}(U).$$

Here's how: require that $d$ be zero on the basis vectors

$$1, dx, dy, dz, dy \wedge dz, dz \wedge dx, dx \wedge dy, \ldots$$

and that $d$ be $\mathbb{R}$-linear

$$i.e. \quad d(aw + bw) = adw + bdw, \quad etc.$$ 

and satisfy the product rule

$$d(f\alpha) = df \wedge \alpha + f \, d\alpha.$$ 

So for example

$$d(f \, dx) = \frac{\partial f}{\partial x} \, dx + f \, d(dx)$$

$$= (f_x \, dx + f_y \, dy + f_z \, dz) \wedge dx.$$
\[ \begin{align*}
&= f_x \, dx \wedge dx + f_y \, dy \wedge dx + f_z \, dz \wedge dx \\
&= - f_y \, dx \wedge dx \quad \text{by "swapping dx's"}
\end{align*} \]

\[ \Rightarrow \ dx \wedge dx = 0 \]

\[ = f_z \, dz \wedge dx - f_y \, dx \wedge dy , \]

\begin{align*}
\text{and} \\
\begin{align*}
\left( f \, dx \wedge dy \right) & \quad \text{2-form} \\
\mathrm{d} \left( f \, dx \wedge dy \right) & = df \wedge dx \wedge dy + f \, \mathrm{d} \left( dx \wedge dy \right) \\
& = f_x \, dx \wedge dx \wedge dy + f_y \, dy \wedge dx \wedge dy + f_z \, dz \wedge dx \wedge dy \\
& = 0 \\
& = f_z \, dz \wedge dx \wedge dy \\
\text{3-form}
\end{align*}
\end{align*}

(Clearly \( \mathrm{d} \left( f \, dx \wedge dy \wedge dt \right) = 0 \).)

Now notice that

\[ \Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \]

\[ f \quad \mapsto \quad df = f_x \, dx + f_y \, dy + f_z \, dz \]

"matches" the gradient \( \nabla f = (f_x, f_y, f_z) \),

\[ \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \]

\[ P \, dx + Q \, dy + R \, dz \quad \mapsto \quad \left( P_y \, dy \wedge dx + P_z \, dz \wedge dx \right) \\
\left( Q_x \, dx \wedge dy + Q_z \, dz \wedge dy \right) \\
\left( R_x \, dx \wedge dz + R_y \, dy \wedge dz \right) \\
= \left( R_y - Q_z \right) dy \wedge dz + \left( P_z - R_x \right) dz \wedge dx + \left( Q_x - P_y \right) dx \wedge dy \\
\] "matches" \( \text{curl} \, \left( P, Q, R \right) = \left( R_y - Q_z, P_z - R_x, Q_x - P_y \right) \), and
\[ \mathcal{L}^2(U) \xrightarrow{d} \mathcal{L}^3(U) \]

\[
p \text{dy} + q \text{dx} + r \text{dz} \rightarrow p_x \text{dxdyndz} + q_y \text{dyndzdx} + r_z \text{dzdxndy} = \text{dxdyndz} = \text{dyndzdx} = \text{dzdxndy} \]

\[= (p_x + q_y + r_z) \text{dxdyndz} \]

"matched" \( \text{div}(p,q,r) = p_x + q_y + r_z \).

So the identity \( \text{curl} (\nabla f) = 0 \) becomes

\[ d(df) = 0 , \]

and the identity \( \text{div}(\text{curl}(F)) = 0 \) becomes

\[ d(d\omega) = 0. \quad (\omega \in \mathcal{L}^1(U)) \]

In all degrees we therefore have \( d \circ d = 0 \).

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**The "generalized Stokes theorem"**

We already know how to integrate differential forms:

- \( \omega \in \mathcal{L}^0(U) \), \( \Gamma = \{ x \} \) \( \Rightarrow \int_{\{ x \}} \omega := f(x) \),

  \( \text{a point in } U \)

- \( \omega \in \mathcal{L}^1(U) \), \( \Gamma = [p,q] \) \( \Rightarrow \int_{[p,q]} \omega := \int_{\Gamma} \omega := f(p) - f(q) \).

  \( \text{a formal difference of points} \)

- \( \omega \in \mathcal{L}^2(U) \), \( \Gamma = \partial \Sigma \) \( \Rightarrow \int_{\partial \Sigma} \omega := \int_{\Gamma} \omega := \int_{\Gamma} p \text{dxdy} + q \text{dydz} + r \text{dzdx} \)

  \( \text{oriented curve} \)

- \( \omega \in \mathcal{L}^3(U) \), \( \Gamma = \Sigma \) \( \Rightarrow \int_{\Sigma} \omega := \int_{\Sigma} p \text{dxdyndz} + q \text{dyndzdx} + r \text{dzdxndy} \)

  \( \text{oriented surface} \)
\[ \omega \in \Omega^3(U), \quad \Gamma = \partial V \implies \int_{\Gamma} \omega = \int_{\partial V} f \, dS \, \text{dA} \, \text{dV} \]

Write "\( \partial \)" for the boundary operator, \( \text{v.i.} \)

\[
\Gamma : \begin{array}{c}
\begin{array}{c}
\partial T := \partial \{ \mathcal{F} \} = \mathcal{F} - \mathcal{G} \\
\partial \Gamma := \partial \{ \mathcal{F} \} = \mathcal{F} \\
\partial \Gamma := \partial \{ \mathcal{F} \} = \mathcal{F}
\end{array}
\end{array}
\]

**Theorem:** Let \( \Gamma \) be a \((k+1)\)-dimensional "object", with \( \partial \Gamma \) as its \( k \)-dim boundary; \( \mathcal{F} \in \Omega^k(U) \) a \( k \)-form. Then

\[ \int_{\Gamma} \omega = \int_{\partial \Gamma} \omega. \]

**Proof:**

\[
\begin{align*}
\int_{\Gamma} \omega & = \int_{\partial \Gamma} \omega + \int_{\mathcal{F}} f(x, y, z) \, dx + f(x, y, z) \, dy + f(x, y, z) \, dz \\
& = \int_{\mathcal{F} \in \Gamma} f(x, y, z) \, dx + f(x, y, z) \, dy + f(x, y, z) \, dz
\end{align*}
\]

**Poincaré–Gabriel Laplace equation**

\[ \int_{\mathcal{F}} \nabla f \cdot \mathbf{d}S = \text{P.T. for line integrals} \]

\(^\dagger\) [precise] smooth curve, surface, or solid, with [precise] smooth boundary (the technical term is "manifold with boundary")
"Holeyness"?

Now consider the sequence of d's

(1) \( \mathbb{R} \rightarrow \mathcal{L}^0(U) \rightarrow \mathcal{L}^1(U) \rightarrow \mathcal{L}^2(U) \rightarrow \mathcal{L}^3(U) \rightarrow \mathbb{0} \)

and recall that \( d \circ d = 0 \).

- If \( U \) is connected, then \( \nabla f = \vec{0} \Rightarrow f \) is constant.
  That is: for \( \omega \in \mathcal{L}^0(U) \), \( d\omega = 0 \Rightarrow \omega \) constant.
• If $U$ is simply connected ("1-connected"), then $\text{curl}(F) = 0 \implies F = \nabla f$ for some $f$. Here that translates to:
  for $\omega \in \Omega^1(U)$, $d\omega = 0 \implies \omega \in \mathcal{L}^0(U)$.

• If $U$ is 2-connected, $\text{div}(F) = 0 \implies F = \text{curl} \mathcal{G}$ for some $\mathcal{G}$. Here this becomes: for $\omega \in \Omega^2(U)$,
  $d\omega = 0 \implies \omega \in \mathcal{L}^2(U)$.

• Finally, one can show that any function $f$ is $\text{div}(F)$ for some $F$. Here this means that $d_2$ is surjective.

The upshot is that if $U$ is convex (hence 0, 1, and 2-connected), then $\ker(d_k) = \text{image}(d_{k-1})$ for $k = 0, 1, 2, \& 3$. This is called the "Poincaré Lemma", and makes (*) into what is called an exact sequence.

Conversely, the failure of "exactness of (*)" can be used as a measure of the "holeyness" of $U$:

- $H^0(U) := \ker(d_0)$ measures the # of connected components of $U$ (if $U$ is connected).
- $H^1(U) := \frac{\ker(d_1)}{\text{im}(d_0)}$ measures the # of "tunnels" through $U$ (if $U$ is convex, these are zero).
- $H^2(U) := \frac{\ker(d_2)}{\text{im}(d_1)}$ measures the # of "solid holes" in $U$.
Corollary: If $\omega \in \Omega^k(U)$ has $\partial \omega = 0$, $\gamma$ is a closed k-chain object ($\partial \gamma = 0$), and $\int_{\gamma} \omega \neq 0$, then $\text{H}^k(U) \neq \{0\}$.

Proof: If $\text{H}^k(U) = 0$, then $\partial \omega = 0 \implies \omega = d\gamma$ for some $\gamma \in \Omega^{k-1}(U)$. The Theorem then implies $\int_{\gamma} \omega = \int_{\gamma} d\gamma = \int_{\partial \gamma} \gamma = 0$, a contradiction.

Example: $k = 2$, $U = \mathbb{R}^3 \setminus \{(0,0,0)\}$, $\gamma = \text{Sphere}$, and

$$\omega = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}.$$

We showed (in effect): $\int_{\gamma} \omega = 4\pi$. So $\text{H}^2(U) \neq \{0\}$. //

Everything we have done here extends to higher dimension and to spaces ("manifolds") other than open subsets of $\mathbb{R}^n$. In the last couple of classes I will give some applications.