Lecture 51 : Some applications

Homotopic maps

These are maps which can be smoothly deformed into each other. This has a powerful connection with integrals of differential forms.

<u>Definition</u>: Let $M \subset \mathbb{R}^m$ be a compact k-manifold with $\partial M = \emptyset$, and $X \subset \mathbb{R}^n$ any manifold. Two C^∞ maps

$$\vec{f}, \vec{g} : M \to X$$

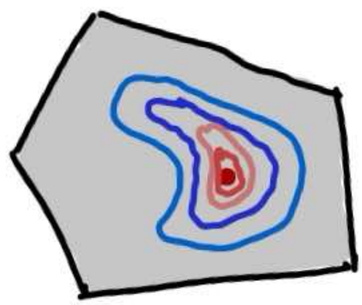
are called <u>homotopic</u> (written $\vec{f} \simeq \vec{g}$) if there exists a C^∞ map

$$\vec{H} : M \times [0,1] \to X$$

with $\vec{H}(\vec{\mu}, 0) = f(\vec{\mu})$ and $\vec{H}(\mu, 1) = g(\mu)$.

Ex/ If X is convex, <u>any</u> two maps into it are homotopic : put $\vec{H}(\vec{\mu}, t) := t\,\vec{g}(\vec{\mu}) + (1-t)\vec{f}(\vec{\mu})$.

e.g. any map from S^1 (= circle) into a convex X can be shrunk to a constant map (image = point). We <u>cannot</u> do this for maps from S^1 into (say) $X = \mathbb{R}^2 \setminus \{0\}$, because $\vec{H}$ has to stay inside X and the loop may <u>not</u> be moved across the origin. //



A form ω is called closed if $d\omega = 0$ and exact if it is d of something. (Exact forms are closed.)

Proposition: If $\vec{f} \simeq \vec{g}$, and $\omega \in \Omega^k(X)$ is closed, then $\int_M \vec{f}^* \omega = \int_M \vec{g}^* \omega$.

Proof: $$0 = \int_{M \times [0,1]} \vec{H}^*(d\omega) = \int_{M \times [0,1]} d(\vec{H}^* \omega) \overset{\text{Stokes}}{=} \int_{\partial(M \times [0,1])} \vec{H}^* \omega$$
$$= \int_{M \times \{1\}} \vec{H}^* \omega - \int_{M \times \{0\}} \vec{H}^* \omega = \int_M \vec{g}^* \omega - \int_M \vec{f}^* \omega. \qquad \square$$

Corollary: If X is simply connected — i.e. any "loop" $\vec{f} : S^1 \longrightarrow X$ is homotopic to a constant map — then every closed 1-form ω on X is exact.

Proof: Given a closed curve $\mathcal{C}$ on X, the parametrizing map $\vec{f} : S^1 \to X$ is (by assumption) homotopic to a constant map $\vec{g}$. Since $\vec{g}^* \omega = 0$, the Proposition gives
$$\oint_{\mathcal{C}} \omega = \int_{S^1} \vec{f}^* \omega = \int_{S^1} \vec{g}^* \omega = 0.$$

It follows that $\omega = df$ for some function f on X. (Namely, fix a point $p \in X$ and let $f(q) = \int_p^q \omega$.) $\square$

We turn to two applications, one geometric and one algebraic.

#1 : Why you shouldn't be a hairstylist to a sphere

A C^∞ vector field on the unit sphere S^2 is a C^∞ map $\vec{v}: S^2 \to \mathbb{R}^3$ such that $\vec{v}(\vec{x}) \perp \vec{x}$ for each $\vec{x} \in S^2$. (That is, $\vec{v}(\vec{x})$ lies in the tangent plane $T_{\vec{x}} S^2$.)

<u>Hairy sphere theorem</u>: Any C^∞ vector field on S^2 must be zero somewhere. (So a unit length vector field on S^2 can't be smooth — the hairstyle will have a "singularity".)

<u>Proof</u>: Suppose $\vec{v}$ has no zero on S^2. Then we can define $\hat{v}(\vec{x}) := \dfrac{\vec{v}(\vec{x})}{\|\vec{v}(\vec{x})\|}$, hence a map $\hat{v}: S^2 \to S^2$.

Let $\vec{f}: S^2 \to S^2$ be the identity $(\vec{f}(\vec{x}) = \vec{x})$

$\vec{g}: S^2 \to S^2$ be the antipodal map $(\vec{g}(\vec{x}) = -\vec{x})$

and set

$$\vec{H}(\vec{x},t) := (\cos \pi t)\, \vec{x} \overset{\perp}{+} (\sin \pi t)\, \hat{v}(\vec{x}).$$

This has $\vec{H}(\vec{x},0) = \vec{x} = f(\vec{x})$, $\vec{H}(\vec{x},1) = -\vec{x} = \vec{g}(\vec{x})$, and

$\|\vec{H}(\vec{x},t)\|^2 = (\cos \pi t)^2 \underbrace{\|\vec{x}\|^2}_{1} + (\sin \pi t)^2 \underbrace{\|\hat{v}(\vec{x})\|^2}_{1} = 1$ by the Pythagorean theorem. So $\vec{H}(\vec{x},t) \in S^2$ for all $\vec{x} \in S^2$ & $t \in [0,1]$, hence $\vec{H}$ gives a homotopy between $\vec{f}$ & $\vec{g}$. Applying the $\boxed{\text{Proposition}}$ to the area form $(\omega =)\ \sigma = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$

$$\Rightarrow 4\pi = \int_{S^2} \sigma = \int_{S^2} \vec{f}^* \sigma \boxed{=} \int_{S^2} \vec{g}^* \sigma \underset{\text{why?}}{=} \int_{S^2} -\sigma = -\int_{S^2} \sigma = -4\pi \quad \times$$

$\square$

#2: Why every polynomial has a (possibly complex) root

Let $p(z) = z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_1 z + \alpha_0$ be a polynomial with $\alpha_i \in \mathbb{C}$, in a complex variable z. (We shall identify $\mathbb{C}$ with $\mathbb{R}^2$, via $z = x + iy \leftrightarrow (x,y)$.)

<u>Fundamental Theorem of Algebra:</u> p has a root in $\mathbb{C}$.

(Of course, if $p(\alpha)=0$, then we can factor out a $(z-\alpha)$, and $p(z)/(z-\alpha)$ has a root too; continuing, we can factor p into linear factors.)

<u>Proof</u>: Take $R>0$ sufficiently large that $|z|=R \Rightarrow$

$$\left| \frac{p(z) - z^n}{z^n} \right| = \left| \frac{\alpha_{n-1}}{z} + \dots + \frac{\alpha_0}{z^n} \right| \leq \frac{1}{2}.$$

Write $B := \{ z \in \mathbb{C} \mid |z| \leq R \}$. (large disk; ∂B = circle of radius R)

Define $H : \partial B \times [0,1] \to \mathbb{C} \setminus \{0\}$ by

$$H(z,t) := t z^n + (1-t)\, p(z),$$

giving a homotopy between $g(z) = z^n$ and $p(z)$ as functions from ∂B to $\mathbb{C}\setminus\{0\}$.

[Check: we need that H doesn't hit 0. Write

$$|H(z,t)| = |z^n + (1-t)(p(z) - z^n)| \geq |z^n| - (1-t)|p(z) - z^n|$$

$$\geq |z^n| - (1-t)\frac{|z^n|}{2} = \left(\frac{1}{2} + \frac{t}{2}\right)|z^n| \underset{z \in \partial B}{=} \left(\frac{1}{2} + \frac{t}{2}\right) R^n$$

$$\geq R^n/2 > 0. \ ]$$

Now $\omega = \dfrac{-y\,dx + x\,dy}{x^2+y^2} \in \Omega^1(\underbrace{\mathbb{R}^2\setminus\{\vec{0}\}}_{=\,\mathbb{C}\setminus\{0\}})$ is closed

and $g(z) = z^n$ for $z \in \partial B$ translates (think $z = Re^{it}$) to

$$g(R\cos t, R\sin t) = \left(R^n\cos(nt),\, R^n\sin(nt)\right).$$

So

$$\int_{\partial B} g^*\omega = \int_0^{2\pi} (n\sin^2 nt + n\cos^2 nt)\,dt = 2\pi n$$

$\Vert$ Proposition

$\int_{\partial B} p^*\omega$, and if p has no root in B then p maps $B \to \mathbb{C}\setminus\{0\}$ and Stokes's

$\Vert$ Stokes $\leftarrow$ Theorem applies ...

... leading to the contradiction shown.

$$\int_B d(p^*\omega) = \int_B p^*(\underbrace{d\omega}_{=0}) = 0 \quad \lightning \qquad \square$$