Lecture 6 : Coordinates

Today we discuss coordinates with respect to a basis: this turns abstract vectors into column vectors (in $\mathbb{R}^n$) and abstract transformations into matrices.

Let V be a vector space (of dim. n) and $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ a basis of V. Given a vector $\vec{v} \in V$, we can write $\vec{v} = \sum_{i=1}^n \underbrace{v_i}_{\in \mathbb{R}} \vec{b}_i$ in exactly one way. Define the coordinate vector (with resp. to $\mathcal{B}$)

$$[\vec{v}]_{\mathcal{B}} := \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n.$$

<u>Proposition 1</u>: $[\cdot]_{\mathcal{B}} : V \to \mathbb{R}^n$ is a (linear) isomorphism.

<u>Proof</u>: If $\vec{w} = \sum w_i \vec{b}_i$ and $\vec{v} = \sum v_i \vec{b}_i$, then $\alpha\vec{w} + \beta\vec{v} = \sum (\alpha w_i + \beta v_i)\vec{b}_i$ and so

$$[\alpha\vec{w} + \beta\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha w_1 + \beta v_1 \\ \vdots \\ \alpha w_n + \beta v_n \end{pmatrix} = \alpha \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} + \beta \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \alpha[\vec{w}]_{\mathcal{B}} + \beta[\vec{v}]_{\mathcal{B}}.$$

That is,

$[\cdot]_{\mathcal{B}}$ is linear. It is 1-1 b/c $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow$ $\vec{v} = 0\vec{b}_1 + \cdots + 0\vec{b}_n = \vec{0}$, and onto b/c any given $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ is the coord. vector of $\alpha_1\vec{b}_1 + \cdots + \alpha_n\vec{b}_n$. $\square$

Ex/ Let $\mathcal{B} = \{1, t, \ldots, t^k\}$ be the basis of $\mathbb{P}_k$.

Then $[a_0 + a_1 t + \cdots + a_k t^k]_{\mathcal{B}} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_k \end{pmatrix}$. //

Now let W be another vector space, of dimension m, with basis $\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_m\}$; and $T: V \to W$ a linear transformation ($T \in \mathcal{L}(V, W)$).

The (coordinate) matrix of T (w.r.t. $\mathcal{B}$ & $\mathcal{C}$) is

$$_{\mathcal{C}}[T]_{\mathcal{B}} := \begin{pmatrix} \uparrow & & \uparrow \\ [T\vec{b}_1]_{\mathcal{C}} & \cdots & [T\vec{b}_n]_{\mathcal{C}} \\ \downarrow & & \downarrow \end{pmatrix} \in M_{m,n}$$

That is, if $T\vec{b}_j = \sum_{i=1}^{m} t_{ij}\vec{c}_i$ $(\forall j)$ then t_{ij} is the $(i,j)^{\text{th}}$ matrix entry $(\forall i, j)$.

<u>Proposition 2</u>: $_{\mathcal{C}}[\cdot]_{\mathcal{B}} : \mathcal{L}(V, W) \to M_{m,n}$ is an isomorphism of vector spaces.

<u>Proof</u>: (Linearity is left to you.) It is 1-1 b/c if ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ is the zero matrix, $T\vec{b}_j = \sum 0\,\vec{c}_i = \vec{0}\ (\forall j)$ $\Rightarrow$ T is the zero transformation. It is onto b/c <u>given</u> the matrix (t_{ij}), you can <u>define</u> a linear transformation by $T\vec{b}_j := \sum_i t_{ij}\vec{c}_i$ (telling each basis vector where to go). $\square$

Ex/

$$V = \text{span}\{\overset{\vec{b}_1}{\cos(t)}, \overset{\vec{b}_2}{\sin(t)}, \overset{\vec{b}_3}{t\cos(t)}, \overset{\vec{b}_4}{t\sin(t)}\} \subset C^0(\mathbb{R})$$

$$W = \mathbb{P}_4 \text{ with basis } \mathcal{C} = \{\overset{\vec{c}_1}{1}, \overset{\vec{c}_2}{t}, \overset{\vec{c}_3}{t^2}, \overset{\vec{c}_4}{t^3}, \overset{\vec{c}_5}{t^4}\}.$$

$T: V \to W$ sends $f \mapsto \{$degree-4 Taylor polynomial at 0 for $f\}$.

$${}_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{6} & -\frac{1}{2} & 0 \\ \frac{1}{24} & 0 & 0 & -\frac{1}{6} \end{pmatrix}$$

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Ex/ $V = \mathbb{P}_3$, $T = \frac{d}{dt} : V \to V$, $\mathcal{B} = \{1, t, t^2, t^3\}$

$$[T]_{\mathcal{B}} := {}_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

notation

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Ex/ When $V = \mathbb{R}^n$ & $W = \mathbb{R}^m$

$$\mathcal{B} = \{\vec{e}_1, \dots, \vec{e}_n\} \quad \mathcal{C} = \{\vec{e}_1, \dots, \vec{e}_m\}$$

the standard basis is usually written $\underline{\underline{\mathcal{E}}}$ in what follows.

and $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$\vec{x} \mapsto A\vec{x}$ for $A \in M_{m,n}$, we have

simply ${}_{\mathcal{C}}[T]_{\mathcal{B}} = A$. This is because the

columns of A are precisely $A\vec{e}_1, \dots, A\vec{e}_n$ (written w.r.t. the standard basis).

Some basic examples: $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & -1 \end{pmatrix}$ gives

a transformation from $\mathbb{R}^3 \to \mathbb{R}^2$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - 3y + z \\ x - z \end{pmatrix}.$$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ reflects about the x-axis

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ -y \end{pmatrix}$$

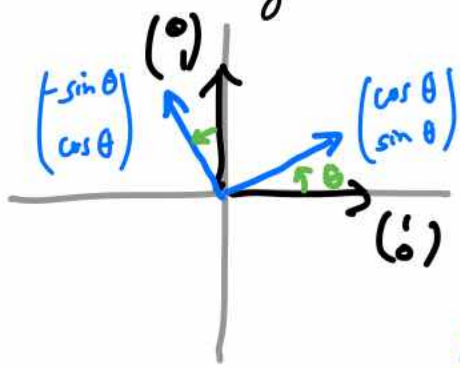
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathbb{R}^3 \mapsto \mathbb{R}^3$ projects to the xy-plane

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$ rotates counter-clockwise by θ

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



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Properties:

① ${}_{\mathcal{C}}[T]_{\mathcal{B}}\,[\vec{v}]_{\mathcal{B}} = [T\vec{v}]_{\mathcal{C}}$ (in $\mathbb{R}^m$)

$m \times n \quad n \times 1 \quad m \times 1$

Proof: If $\vec{v} = \sum_{j=1}^{n} v_j \vec{b}_j$, $T\vec{b}_j = \sum_{i=1}^{m} t_{ij} \vec{c}_i$, then

$$T\vec{v} = T\sum_j v_j \vec{b}_j \underset{\text{linearity}}{\overset{\uparrow}{=}} \sum_j v_j T\vec{b}_j = \sum_j v_j \sum_i t_{ij} \vec{c}_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} t_{ij} v_j\right)\vec{c}_i$$

$\Rightarrow$ ith entry of RHS $= \sum_j t_{ij} v_j =$ ith entry of LHS. □

② Writing $A = {}_{\mathcal{C}}[T]_{\mathcal{B}}$, ① has the 2 consequences:

ⓐ The span of A's columns is $[\mathrm{Im}(T)]_{\mathcal{C}}$ $\Rightarrow$

TFAE —

- T is onto
- columns of A span $\mathbb{R}^m$
- rref(A) has no rows of all 0's

ⓑ Null space of A (sol'ns of $A\vec{x} = \vec{0}$) is $[\mathrm{Ker}(T)]_{\mathcal{B}}$ $\Rightarrow$

TFAE —

- T is 1-1
- columns of A are independent
- rref(A) has a leading 1 in every column.

If $\dim V = \dim W$, then: the 2 statements about rref(A) are equivalent to $\mathrm{rref}(A) = \mathbb{I}_n$, hence to invertibility of A (Lecture 5); and the 2 statements about T are equivalent (to each other and) to T being an isomorphism/invertible. So we get...

(3) TFAE — • T is an isomorphism/invertible

• $\mathrm{rref}(A) = \mathbb{I}_n$

• A is invertible.

(4) $\mathrm{rank}(T) = \mathrm{rank}(A)$

$\uparrow$ $\dim(\mathrm{Im}(T))$ $\uparrow$ # of pivot columns

Pf: The null space of A is the same as the null space of rref(A), which is spanned by the solutions of $\mathrm{rref}(A)\vec{x} = \vec{0}$ obtained by setting each free variable to 1 (& others to 0)

$\Rightarrow \dim(\text{nullspace}(A)) = n - \#\text{pivots} = n - \mathrm{rank}(A)$.

$$\underset{\dim(\ker(T))}{\overset{\dim(\text{nullspace}(A))}{\|}} \overset{R+N}{=} n - \mathrm{rank}(T). \qquad \square$$

Ex/ Suppose ${}_{\mathcal{E}}[T]_{\mathcal{B}} = \begin{pmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ -1 & 0 & -6 & -2 \end{pmatrix}$. Is T 1-1? onto?

Ex/ Suppose ${}_{\mathcal{E}}[T]_{\mathcal{B}} = \begin{pmatrix} 3 & 4 & 7 & 4 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Is T invertible?

How about $\begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$? (If not, what is its rank?)

(5) Compositions of linear transformations

bases: $\mathcal{A}$ $\qquad$ $\mathcal{B}$ $\qquad$ $\mathcal{C}$

Given
$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend right, "R := T \circ S"'] & V \arrow[r, "T"] & W \end{tikzcd}$$

dims: ℓ $\qquad$ n $\qquad$ m

I claim that

matrix multiplication

$$(*) \qquad \underbrace{{}_{\mathcal{C}}[T]_{\mathcal{B}}}_{m\times n} \cdot \underbrace{{}_{\mathcal{B}}[S]_{\mathcal{A}}}_{n\times \ell} = \underbrace{{}_{\mathcal{C}}[R]_{\mathcal{A}}}_{m\times \ell}.$$

<u>Proof</u>: Start with $S\vec{a}_k = \sum_j s_{jk}\vec{b}_j$, $T\vec{b}_j = \sum t_{ij}\vec{c}_i$,

so
$$\sum_i r_{ik}\vec{c}_i = R\vec{a}_k = T(S\vec{a}_k) = T\Big(\sum_j s_{jk}\vec{b}_j\Big)$$
$$= \sum_j s_{jk} T\vec{b}_j = \sum_j s_{jk}\sum_i t_{ij}\vec{c}_i$$
$$= \sum_i \Big(\sum_j t_{ij}s_{jk}\Big)\vec{c}_i.$$

Since the $\vec{c}_i$ are L.I., the coeffs. must agree:

$$\sum_j t_{ij}s_{jk} = r_{ik} \quad \text{for all } i,k.$$

But this says the LHS & RHS (of $(*)$) matrix entries agree. $\square$

Here are two consequences:

Ex / $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = [T_\theta]_\mathcal{E}$ where

$T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by θ. Clearly

$T_{\theta+\phi} = T_\theta \circ T_\phi$. Taking matrices, we get

$R_{\theta+\phi} = [T_{\theta+\phi}]_\mathcal{E} = [T_\theta]_\mathcal{E} \cdot [T_\phi]_\mathcal{E} = R_\theta R_\phi$.

This recovers the angle addition formulas! —

$$\begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$= \left(\begin{array}{c|c} \cos\theta\cos\phi - \sin\theta\sin\phi & -\sin\theta\cos\phi - \cos\theta\sin\phi \\ \hline \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{array}\right).$$

We also get $R_\theta^n = R_{n\theta}$, with no effort! //

Ex / If T is invertible and $A = {}_\mathcal{C}[T]_\mathcal{B}$,

then ${}_\mathcal{B}[T^{-1}]_\mathcal{C} \, {}_\mathcal{C}[T]_\mathcal{B} = {}_\mathcal{B}[T^{-1} \circ T]_\mathcal{B} = [\mathrm{Id}_V]_\mathcal{B} = I_n$

$\Rightarrow {}_\mathcal{B}[T^{-1}]_\mathcal{C} = A^{-1}$; that is, the inverse matrix is the matrix of the inverse transformation. //