Lecture 7: Change of basis

Let's begin with $\mathbb{R}^n$. Any basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ has associated $n \times n$ matrix

$$P_{\mathcal{B}} := \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & \vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix}.$$

(Recall that an $n \times n$ matrix is invertible $\iff$ its RREF $= \mathbb{I}_n$ $\iff$ columns are independent & span $\mathbb{R}^n$ $\iff$ columns are a basis of $\mathbb{R}^n$. So $\underline{P_{\mathcal{B}} \text{ is invertible}}$.)

If $\vec{x} \in \mathbb{R}^n$ has coordinate vector (w.r.t. $\mathcal{B}$)

$$[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

then (by definition)

$$\underline{\underline{\vec{x}}} = \alpha_1 \vec{b}_1 + \dots + \alpha_n \vec{b}_n = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{b}_1 & \cdots & \vec{b}_n \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \underline{\underline{P_{\mathcal{B}} [\vec{x}]_{\mathcal{B}}}}.$$

In fact, "$\vec{x}$" is itself a coordinate vector — with respect to the standard basis $\mathcal{E} = \{\vec{e}_1, \dots, \vec{e}_n\}$: $\vec{x} = [\vec{x}]_{\mathcal{E}}$.

So we call $P_{\mathcal{B}}$ the change-of-basis matrix (or change of coordinates matrix) from $\mathcal{B}$ to $\mathcal{E}$.

Of course, you don't <u>need</u> a matrix to go from $[\vec{x}]_{\mathcal{B}}$ to $\vec{x}$, but it's essential for going the other way:

Ex/ $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, $\vec{x} = \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.

Have $P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \vec{x}$, so $[\vec{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\vec{x}$; to get $P_{\mathcal{B}}^{-1}$,

row-reduce: $\left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ 4 & 1 & & & 1 & \\ -4 & -4 & 1 & & & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & 1 & & -4 & 1 & \\ & & 1 & -20 & 4 & 1 \end{array}\right]$

$$\Longrightarrow [\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 1 & & \\ -4 & 1 & \\ -20 & 4 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ -63 \end{pmatrix}. \quad //$$

Next let V be a vector space, and $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ & $\mathcal{C} = \{\vec{c}_1, \dots, \vec{c}_n\}$ be two bases. The $n \times n$ matrix

$$\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} := \begin{pmatrix} \uparrow & & \uparrow \\ [\vec{b}_1]_{\mathcal{C}} & \cdots & [\vec{b}_n]_{\mathcal{C}} \\ \downarrow & & \downarrow \end{pmatrix}$$

has the property that if $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$, then

$$\underline{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,[\vec{v}]_{\mathcal{B}}} = \beta_1[\vec{b}_1]_{\mathcal{C}} + \cdots + \beta_n[\vec{b}_n]_{\mathcal{C}}$$
$$= [\beta_1\vec{b}_1 + \cdots + \beta_n\vec{b}_n]_{\mathcal{C}} = \underline{[\vec{v}]_{\mathcal{C}}}.$$

This means that

$$\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}[\vec{v}]_{\mathcal{C}} = [\vec{v}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}\,[\vec{v}]_{\mathcal{C}} \quad \text{for all } \vec{v}\in V,$$

so that $\left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$. Given another basis $\mathcal{A}$, we also have that $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\cdot\underset{\mathcal{B}\leftarrow\mathcal{A}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{A}}{P}$.

Ex/ If $V=\mathbb{R}^n$ then $P_{\mathcal{B}} = \underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ and $P_{\mathcal{C}} = \underset{\mathcal{E}\leftarrow\mathcal{C}}{P}$.

So $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\cdot\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = P_{\mathcal{C}}^{-1}\cdot P_{\mathcal{B}}$ gives you a way to calculate $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. //

Ex/ Write $p(t) = 1+5t-2t^2$ with respect to the basis $\mathcal{B} = \{1, t-1, (t-1)^2\}$ of $\mathbb{P}_2$.

$$\underset{t\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \underset{\text{inverse}}{\overset{\text{take}}{\Longrightarrow}} \underset{\mathcal{B}\leftarrow t}{P} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

The basis $\{1, t, t^2\}$

$$\Rightarrow [p(t)]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow t}{P}\,[p(t)]_t = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow p(t) = 4\cdot\underline{1} + 1\cdot\underline{(t-1)} + (-2)\,\underline{(t-1)^2}$$ //

Next let $T: V \to W$ be a linear transformation and $\mathcal{A}, \mathcal{B}$ bases of V, $\mathcal{C}, \mathcal{D}$ bases of W.

<u>Claim</u>: $${}_{\mathcal{C}}[T]_{\mathcal{B}} = \underset{\mathcal{C}\leftarrow\mathcal{D}}{P} \cdot {}_{\mathcal{D}}[T]_{\mathcal{A}} \cdot \underset{\mathcal{A}\leftarrow\mathcal{B}}{P}.$$

<u>Proof</u>: $$\underline{{}_{\mathcal{C}}[T]_{\mathcal{B}}}\,[\vec{v}]_{\mathcal{B}} = [T\vec{v}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{D}}{P}\,[T\vec{v}]_{\mathcal{D}}$$

any $\vec{v} \in V$

$$= \underset{\mathcal{C}\leftarrow\mathcal{D}}{P}\,{}_{\mathcal{D}}[T]_{\mathcal{A}}\,[\vec{v}]_{\mathcal{A}} = \underline{\underset{\mathcal{C}\leftarrow\mathcal{D}}{P}\,{}_{\mathcal{D}}[T]_{\mathcal{A}}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{P}}\,[\vec{v}]_{\mathcal{B}}.$$

$\square$

This is most frequently used for $T: V \to V$ to write
$$[T]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{A}}{P}\,[T]_{\mathcal{A}}\,\underset{\mathcal{A}\leftarrow\mathcal{B}}{P}.$$

inverse

Ex/ In the special case of a matrix transformation
$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n \qquad (A = n\times n \text{ matrix}),$$
$$\vec{x} \longmapsto A\vec{x}$$
we have $A = [T]_{\mathcal{E}}$. How do we write T with respect to $\mathcal{B}$? Apply the above:

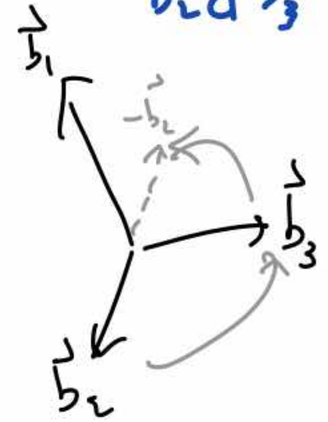
$$[T]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{E}}{P}\,[T]_{\mathcal{E}}\,\underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = P_{\mathcal{B}}^{-1}\cdot A\cdot P_{\mathcal{B}}.$$

For instance, if $n=2$, $\mathcal{B} = \left\{\begin{pmatrix}2\\-1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right\}$, and

$A = \begin{pmatrix}3 & 4\\-1 & -1\end{pmatrix}$, then $P_{\mathcal{B}} = \begin{pmatrix}2 & 1\\-1 & 2\end{pmatrix} \Rightarrow P_{\mathcal{B}}^{-1} = \frac{1}{5}\begin{pmatrix}2 & -1\\1 & 2\end{pmatrix}$

$$\Rightarrow [T]_{\mathcal{B}} = \frac{1}{5}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\\ -1 & -1\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix} = \begin{pmatrix}1 & 5\\ 0 & 1\end{pmatrix}.$$ //

Ex/ Find a matrix (w.r.t. standard basis!) for rotating $\mathbb{R}^3$ $90°$ about the axis spanned by $\vec{b}_1 = \begin{pmatrix}1\\2\\2\end{pmatrix}$, given that $\vec{b}_1$, $\vec{b}_2 = \begin{pmatrix}-2\\2\\-1\end{pmatrix}$, and $\vec{b}_3 = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$ are orthogonal and $\vec{b}_2$ & $\vec{b}_3$ are of the same length.



Clearly we must have
$$\vec{b}_1 \mapsto \vec{b}_1$$
$$\vec{b}_2 \mapsto \vec{b}_3$$
$$\vec{b}_3 \mapsto -\vec{b}_2$$

so $[T]_{\mathcal{B}} = \begin{pmatrix}1 & & \\ & & -1\\ & 1 & \end{pmatrix} \Rightarrow$

$$A = [T]_{\mathcal{E}} = P_{\mathcal{B}}[T]_{\mathcal{B}}P_{\mathcal{B}}^{-1} = \begin{pmatrix}1 & -2 & -2\\ 2 & 2 & -1\\ 2 & -1 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}\cdot\frac{1}{9}\begin{pmatrix}1 & 2 & 2\\ -2 & 2 & -1\\ -2 & -1 & 2\end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix}1 & -4 & 8\\ 8 & 4 & 1\\ -4 & 7 & 4\end{pmatrix}.$$

Not a matrix one would just "guess"! //

Ex/ On the other hand, we often want to change basis to make the matrix <u>simpler</u>, not more complicated. If we can choose the bases on <u>both</u> sides <u>independently</u>, then we can always make the matrix of T very simple.

Let $T : \underset{\dim. n}{V} \to \underset{\dim. m}{W}$ be of rank r, and pick

- $\vec{w}_1, \dots, \vec{w}_r$ a basis of $\mathrm{Im}(T) \subseteq W$
  $\rightsquigarrow$ complete to basis $\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_r; \vec{w}_{r+1}, \dots, \vec{w}_m\}$ of W
- $\vec{v}_1, \dots, \vec{v}_r$ vectors sent by T to $\vec{w}_1, \dots, \vec{w}_r$
- $\vec{v}_{r+1}, \dots, \vec{v}_n$ basis of $\ker(T)$ (know there are $n-r$ since rank + nullity $= n$)

If $\sum_{i=1}^{r} a_i \vec{v}_i + \sum_{i=r+1}^{n} b_i \vec{v}_i = \vec{0}$, then applying T gives

$$\sum_{i=1}^{r} a_i \vec{w}_i + \vec{0} = \vec{0} \implies a_i = 0$$

$$\implies \sum_{i=r+1}^{n} b_i \vec{v}_i = \vec{0} \implies b_i = 0.$$

So the $\{\vec{v}_i\}$ are independent hence a basis $\mathcal{B}$ of V,

and
$$_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{array}{c} \left(\begin{array}{c|c} \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \begin{array}{l} \updownarrow r \\ \updownarrow m-r \end{array} \\ \begin{array}{cc} \xleftrightarrow{\;r\;} & \xleftrightarrow{\;n-r\;} \end{array} \end{array}.$$

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