Lecture 8: Determinants (I)

Suppose you were looking for a function

$$\det : \left\{ \begin{matrix} n \times n \text{ real} \\ \text{matrices} \end{matrix} \right\} \longrightarrow \mathbb{R}$$

with the following 3 properties, where we think of an $n \times n$ matrix A as an ordered n-tuple of row vectors $\begin{pmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{pmatrix}$:

(i) <u>Multilinearity</u>: $\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ a\vec{r}_i + b\vec{s}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} = a \cdot \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix} + b \cdot \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{s}_i \\ \vdots \\ \vec{r}_n \end{pmatrix}$

(ii) <u>Antisymmetry</u>: $\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_n \end{pmatrix} = -\det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_j \\ \vdots \\ \vec{r}_i \\ \vdots \\ \vec{r}_n \end{pmatrix}$

(iii) <u>Normalization</u>: $\det \begin{pmatrix} \vec{e}_1 \\ \vdots \\ \vec{e}_n \end{pmatrix} = 1$.

MAIN THEOREM For each n, such a function <u>exists</u> and <u>is unique</u>!

Anticipating this result, we'll call this function <u>the determinant</u>, written $\det(A)$ or $|A|$.

<u>Theorem 1</u>: If 2 rows of A are equal, $\det A = 0$.

<u>Proof</u>: If $\vec{r}_i = \vec{r}_j$, then by (ii), $\det A = -\det A$

$$\Rightarrow 2 \det A = 0 \Rightarrow \det A = 0. \qquad \square$$

$\boxed{2\times 2 \text{ matrices}}$ Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We can use the defining properties (i)–(iii) to compute $\det A$:

Case 1: $a = c = 0$.
$$|A| = \begin{vmatrix} b\vec{e}_2 \\ d\vec{e}_2 \end{vmatrix} \underset{(i)}{=} bd \begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \end{vmatrix} \underset{\text{Thm. 1}}{=} 0.$$

Case 2: $a = 0$.
$$|A| = \begin{vmatrix} b e_2 \\ c e_1 + d e_2 \end{vmatrix} \underset{(i)}{=} bc \begin{vmatrix} \vec{e}_2 \\ \vec{e}_1 \end{vmatrix} + bd \cancelto{0}{\begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \end{vmatrix}}$$
$$= -bc \begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \end{vmatrix} \underset{(iii)}{=} -bc.$$

Case 3: $a \neq 0$.
$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \frac{c}{a}\underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{0} \underset{(i)}{=} \begin{vmatrix} a & b \\ c & d \end{vmatrix} - \begin{vmatrix} a & b \\ c & \frac{bc}{a} \end{vmatrix}$$
$$\underset{(i)}{=} \begin{vmatrix} a & b \\ 0 & d - \frac{bc}{a} \end{vmatrix} \underset{(i)}{=} \begin{vmatrix} a & 0 \\ 0 & d - \frac{bc}{a} \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & b \\ 0 & d - \frac{bc}{a} \end{vmatrix}}_{0 \text{ by Case 1}} \underset{(i)}{=} a\left(d - \frac{bc}{a}\right) \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}}_{1 \text{ by (iii)}}$$
$$= ad - bc.$$

In each case, we get $\underline{ad - bc}$.

Theorem 2: If A is an $n\times n$ upper (or lower) triangular matrix, then $\det(A) = a_{11}\cdot a_{22}\cdot \dots \cdot a_{nn}$.

Proof: (will do for 3×3 matrices)
$$|A| = \begin{vmatrix} \alpha & a & b \\ 0 & \beta & c \\ 0 & 0 & \gamma \end{vmatrix} = \begin{vmatrix} \alpha\vec{e}_1 + a\vec{e}_2 + b\vec{e}_3 \\ \beta\vec{e}_2 + c\vec{e}_3 \\ \gamma\vec{e}_3 \end{vmatrix} = \alpha\beta\gamma \cancelto{1}{\begin{vmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}} + \alpha c\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_1 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}} + a\beta\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_2 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}}$$
$$+ ac\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}} + b\beta\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_3 \\ \vec{e}_2 \\ \vec{e}_3 \end{vmatrix}} + bc\gamma \cancelto{0}{\begin{vmatrix} \vec{e}_3 \\ \vec{e}_3 \\ \vec{e}_3 \end{vmatrix}}$$
$$= \alpha\beta\gamma.$$

$\square$

3 × 3 matrices Let $A = \begin{pmatrix} \alpha & \beta & \gamma \\ a & b & c \\ d & e & f \end{pmatrix}$. We have by (i)

$$|A| = \boxed{\alpha \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ a & b & c \\ d & e & f \end{vmatrix}} . \quad (*)$$

Now notice that

$$\begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & b & c \\ d & e & f \end{vmatrix} - a \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ d & e & f \end{vmatrix}}_{=0} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ d & e & f \end{vmatrix} - d \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{vmatrix}$$

by using Thm. 1 and (i) repeatedly.

In this way, $(*)$ becomes

$$\boxed{\alpha \begin{vmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & e & f \end{vmatrix} + \beta \begin{vmatrix} 0 & 1 & 0 \\ a & 0 & c \\ d & 0 & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & 0 & 1 \\ a & b & 0 \\ d & e & 0 \end{vmatrix}} \quad (**)$$

Applying the same steps as for $n=2$ to the bottom 2 rows in $(**)$ gives:

$$\alpha(bf-ec) \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{|\mathbb{I}_3| = 1} + \beta(af-dc) \underbrace{\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{\overset{(ii)}{=} -|\mathbb{I}_3| = -1} + \gamma(ae-bd) \underbrace{\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}}_{= -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-)(-)|\mathbb{I}_3| = +1}$$

$$= \alpha \underbrace{\begin{vmatrix} b & c \\ e & f \end{vmatrix}}_{=:A_{\hat{1}\hat{1}}} - \beta \underbrace{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}_{=:A_{\hat{1}\hat{2}}} + \gamma \underbrace{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}_{=:A_{\hat{1}\hat{3}}},$$

which is the Laplace expansion of $\det(A)$ along the first row.

n × n matrices Let $A_{\hat{i}\hat{j}} :=$ $(n-1)\times(n-1)$ matrix obtained from A by deleting the i^{th} row & j^{th} column.

Theorem 3: For any fixed i,

"Laplace expansion in the i^{th} row"

$$\det(A) = \sum_{j=1}^{n} a_{ij} \cdot \underbrace{(-1)^{i+j} \cdot \det(A_{\hat{i}\hat{j}})}_{=:\, C_{ij} \,=\, (i,j)^{th} \text{ cofactor}} .$$

Ex/ $\begin{vmatrix} 3 & 5 & 0 \\ 2 & 1 & 3 \\ 0 & 2 & 0 \end{vmatrix} = (-1)^{2+1} 2 \begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} = -2 \cdot 9 = -18.$ //

$\boxed{\text{Another approach to } 3\times3 \text{ matrices}}$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^{3} a_{1i}\vec{e}_i \\ \sum_{j=1}^{3} a_{2j}\vec{e}_j \\ \sum_{k=1}^{3} a_{3k}\vec{e}_k \end{vmatrix} = \sum_{i,j,k=1}^{3} a_{1i}a_{2j}a_{3k} \begin{vmatrix} \vec{e}_i \\ \vec{e}_j \\ \vec{e}_k \end{vmatrix}$$

0 unless i, j, k all different!

$$= a_{11}a_{22}a_{33}\begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31}\begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32}\begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{11}a_{23}a_{32}\begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33}\begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31}\begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

(ii): the determinants of the permutation matrices shown are $(-1)^{\#\text{ of row swaps needed to get } I_3}$

$$= \text{"}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}\text{"}$$

(visual mnemonic)

notice that this is also the # of column swaps needed to reach I_3 (you perform the same swaps in the reverse order).

$\Downarrow$

The same definition of determinant but w/ column vectors replacing row vectors, yields the same answer! This suggests

Theorem 3': $\det A = \sum_{i=1}^{n} a_{ij} (-1)^{i+j} \det(A_{\hat{i}\hat{j}})$, for any fixed j;

(Laplace expansion in the j^{th} column)

and

Theorem 4: $\det A = \det A^T$. (We'll prove these in Lect. 9.)

Ex/ $$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{vmatrix} = 3 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= 3\left(\begin{vmatrix} 2 & -1 \\ 3 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \right) - 1 \left(\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \right)$$

$$= 3 \cdot 4 - 1 \cdot 0 = 12.$$

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We now turn to the

Proof of the Main Theorem: Suppose $\det : M_{n,n} \to \mathbb{R}$ is a function satisfying (i)–(iii). A list $\sigma = (\sigma_1, \dots, \sigma_n)$ of numbers in $\{1, \dots, n\}$ is called a permutation if each number occurs once, in which case the sign $\operatorname{sgn}(\sigma) := (-1)^{\{\#\text{ of swaps required to get from } (1,\dots,n) \text{ to } \sigma\}}$.

By (ii) + (iii),

$$\det \begin{pmatrix} \vec{e}_{\sigma_1} \\ \vdots \\ \vec{e}_{\sigma_n} \end{pmatrix} = \begin{cases} \operatorname{sgn}(\sigma), & \text{if } \sigma \text{ is a permutation} \\ 0, & \text{otherwise;} \end{cases}$$

and so by (i)

$$\det(A) = \det \begin{pmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_n \end{pmatrix} = \det \begin{pmatrix} \sum_{\sigma_1=1}^{n} a_{1,\sigma_1} \vec{e}_{\sigma_1} \\ \vdots \quad \vdots \\ \sum_{\sigma_n=1}^{n} a_{n,\sigma_n} \vec{e}_{\sigma_n} \end{pmatrix} = \sum_{\sigma_1} \cdots \sum_{\sigma_n} a_{1,\sigma_1} \cdots a_{n,\sigma_n} \cdot \det \begin{pmatrix} \vec{e}_{\sigma_1} \\ \vdots \\ \vec{e}_{\sigma_n} \end{pmatrix}$$

$$= \sum_{\sigma \text{ permutation}} \left(\prod_{i=1}^{n} a_{i,\sigma_i} \right) \operatorname{sgn}(\sigma).$$

This proves uniqueness.

Existence is now clear as well: it is easy to check that the formula just derived satisfies (i) – (iii) (left to you). $\square$

<u>Proof of Theorem 3:</u> Let $A_{\hat{k}}(\vec{\beta})$ be the matrix obtained by replacing the k^{th} row of A by the row vector $\vec{\beta}$. Then

$$\det\left(A_{\hat{k}}(\vec{e}_\ell)\right) = \det\begin{pmatrix} a_{11} & \cdots & a_{1\ell} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n\ell} & \cdots & a_{nn} \end{pmatrix} = (-1)^{k-1}\det\begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ a_{11} & \cdots & a_{1\ell} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n\ell} & \cdots & a_{nn} \end{pmatrix}$$

k^{th} row

ℓ^{th} column

omit the a_{k*}'s

swap k^{th} row all the way to the top ($(k-1)$ swaps)

argue as in 3×3 case: "replace" ops. don't change det (see next lecture)

$$= (-1)^{k-1}\det\begin{pmatrix} 0 & \cdots & 1 & \cdots & 0 \\ a_{11} & \cdots & 0 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & 0 & \cdots & a_{nn} \end{pmatrix}$$

swap ℓ^{th} column all the way to the left

$$= (-1)^{(\ell-1)+(k-1)}\det\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A_{\widehat{k\ell}} & \\ 0 & & & \end{array}\right)$$

$= \det(A_{\widehat{k\ell}})$ from the formula for det in the last proof

$$= (-1)^{k+\ell}\det(A_{\widehat{k\ell}}).$$

Now if $\vec{r}_k = k^{\text{th}}$ row of A $= \sum_{\ell=1}^{n} a_{k\ell}\,\vec{e}_\ell$, then

$$\det(A) = \sum_{\ell=1}^{n} a_{k\ell}\det\left(A_{\hat{k}}(\vec{e}_\ell)\right) = \sum_{\ell=1}^{n}(-1)^{k+\ell}a_{k\ell}\det(A_{\widehat{k\ell}}). \qquad \square$$