MATH 233 MIDTERM EXAM 2 SOLUTIONS

This exam consists of 12 multiple choice (machine-graded) problems, worth 5 points each (for a total of 60 points), and 2 pages of written (hand-graded) problems, worth a total of 40 points. No 3x5 cards or calculators are allowed.

PART I: MULTIPLE CHOICE PROBLEMS

You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your **ID number** (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit. **Shade in the corresponding boxes below.** Also **print your name at the top of your card.**

(1) If your Prius XLVII spaceship had velocity vector $\vec{v}(t) = \langle 4, 3\cos t, 3\sin t\rangle$ from $t = 0$ to $t = \pi$, compute the length of the arc along which you traveled.

- (A) π
- (B) 2π
- (C) 3π
- (D) 4π
- (E) $\sqrt{4\pi^2+9}$
- (F) $2\sqrt{4\pi^2+9}$
- (G) none of the above

$$\int_0^\pi \|\vec v(t)\|\,dt = \int_0^\pi \sqrt{16+9\underbrace{(\cos^2 t+\sin^2 t)}_{1}}\,dt = \int_0^\pi 5\,dt = \boxed{5\pi}$$

(2) To reparametrize $\vec{r}(t) = \langle e^t \sin t, e^t \cos t, e^t\rangle$ by arclength from $t = 0$, you must replace t by $t(s) =$ _____.

- (A) $\sqrt{3}(e^s - 1)$
- (B) $\sqrt{3}e^s$
- (C) $\ln(s+1)$
- (D) $\ln(\frac{s}{\sqrt{2}}+1)$
- (E) $\ln(\frac{s}{\sqrt{3}}+1)$
- (F) $\ln(\frac{s}{\sqrt{3}})$
- (G) none of the above

$$\vec r'(t) = \langle e^t(\sin t+\cos t), e^t(\cos t-\sin t), e^t\rangle$$

$$\|\vec r'(t)\| = \sqrt{e^{2t}(\sin^2 t+\cos^2 t+\cancel{2\cos t\sin t}) + e^{2t}(\cos^2 t+\sin^2 t-\cancel{2\cos t\sin t}) + e^{2t}} = \sqrt{3e^{2t}} = \sqrt3\, e^t$$

$s = s(t) = \int_0^t \sqrt3\, e^u\,du = \sqrt3(e^t-1)$. Inverting this,

$t(s) = \ln(\frac{s}{\sqrt3}+1)$.

(3) Find [redacted] $\vec{r}(t)$ if $\vec{a}(t) = \langle 0,0,1\rangle$, $\vec{v}(0) = \langle 1,-1,0\rangle$, and $\vec{r}(0) = \langle 0,1,0\rangle$.

- (A) $\langle 0, 1, \frac{t^2}{2}\rangle$
- (B) $\langle t, 1-t, \frac{t^2}{2}\rangle$
- (C) $\langle t, -t, \frac{t^2}{2}\rangle$
- (D) $\langle 1, -1, t\rangle$
- (E) $\langle t, 1-t, t^2\rangle$
- (F) none of the above

$$\int \vec a\,dt \,(= \vec v) = \langle 0,0,t\rangle + \vec C$$

$$\Rightarrow \vec v = \langle 1,-1,t\rangle$$

$$\vec r = \int \vec v\,dt = \langle t,-t,\tfrac{t^2}{2}\rangle + \vec K$$

$$\Rightarrow \vec r = \langle t, 1-t, \tfrac{t^2}{2}\rangle$$

(4) Let $\vec{r}(t) = \langle 1+\frac{t^4}{4}, \frac{\sqrt{2}t^3}{3}, \frac{t^2}{2}\rangle$. What is the radius of the osculating circle at $\vec{r}(\sqrt{2})$?

- (A) 9
- (B) 3
- (C) 1
- (D) $\frac{1}{3}$
- (E) $\frac{1}{9}$
- (F) none of the above

$$\vec T = \frac{\vec r'}{\|\vec r'\|} = \frac{\langle t^3, \sqrt2 t^2, t\rangle}{\sqrt{t^6+2t^4+t^2}} = \frac{\langle t^2, \sqrt2 t, 1\rangle}{t^2+1}$$

$$\vec T' = \frac{\langle 2t, \sqrt2(1-t^2), -2t\rangle}{(t^2+1)^2}$$

$$\kappa = \frac{\|\vec T'\|}{\|\vec r'\|} = \frac{\sqrt2/(t^2+1)}{t(t^2+1)} = \frac{\sqrt2}{t(t^2+1)^2}$$

$$\Rightarrow \kappa(\sqrt2) = \frac{\sqrt2}{\sqrt2(2+1)^2} = \frac19 \Rightarrow R = 9.$$

$\vec{r}(\sqrt{2}) = \langle 2, \frac{4}{3}, 2 \rangle$

(5) With $\vec{r}(t)$ as in problem (4), which is an equation of the osculating plane at $t = \sqrt{2}$?

(A) $6x + 6y + 3z = 26$
(B) $-6x + 3y + 6z = 4$
(C) $3x - 6y + 6z = 26$
(D) $3x - 6y + 6z = 10$
(E) $-6x + 3y + 6z = 0$
(F) $6x + 6y + 3z = 6$
(G) none of the above

$\vec{N} = \frac{\vec{T}'}{\|\vec{T}'\|} = \frac{\langle \sqrt{2}t, 1-t^2, -\sqrt{2}t \rangle}{t^2+1}$

$\vec{B}(\sqrt{2}) = \vec{T}(\sqrt{2}) \times \vec{N}(\sqrt{2}) = \langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle \times \langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \rangle$
$= \langle \frac{-1}{3}, \frac{2}{3}, \frac{-2}{3} \rangle$

so $-\frac{1}{3}(x-2) + \frac{2}{3}(y - \frac{4}{3}) - \frac{2}{3}(z-2) = 0$. Multiply by 9
$\rightsquigarrow 3x - 6y + 6z = 10$

(6) A bee was flying along a helical path $\vec{H}(t) = \langle \cos t, \sin t, 16t \rangle$ (measured in feet). At $t = 12$, it had a heart attack and died instantly. Where did it land (that is, the (x,y)-coordinates when it hit the xy-plane)? [Hint: reset $t = 0$ when the bee dies, and use the acceleration due to gravity $g = 32$ ft/sec^2.]

(A) $(\cos 12, \sin 12)$
(B) $(-4 \sin 12, 4 \cos 12)$
(C) $(\cos 12 + 3 \sin 12, \sin 12 - 3 \cos 12)$
(D) $(\cos 12 - \sin 12, \sin 12 + \cos 12)$
(E) $(\cos 12 - 4 \sin 12, \sin 12 + 4 \cos 12)$
(F) none of the above

$\vec{a} = \langle 0, 0, -32 \rangle$. $H'(t) = \langle -\sin t, \cos t, 16 \rangle$

$(H'(12) =)\ \vec{v}(0) = \langle -\sin 12, \cos 12, 16 \rangle$

$(H(12) =)\ \vec{r}(0) = \langle \cos 12, \sin 12, 16 \cdot 12 \rangle$

So $\vec{v} = \langle -\sin 12, \cos 12, 16 - 32 \cdot t \rangle$

$\vec{r} = \langle \cos 12 - t \sin 12, \sin 12 + t \cos 12, \underbrace{16t - 16t^2 + 16 \cdot 12}_{\text{solve for this} = 0} \rangle$

$\rightsquigarrow \boxed{t = 4}$

(7) What is $\lim_{(x,y)\to(0,0)} \frac{\sin(3x^2+3y^2)}{x^2+y^2}$?

(A) 1
(B) $\frac{1}{3}$
(C) 3
(D) 9
(E) 0
(F) DNE
(G) none of the above

$= \lim_{r \to 0} \frac{\sin 3r^2}{r^2} = \lim_{\mu \to 0} \frac{\sin \mu}{\frac{1}{3}\mu} = 3 \overbrace{\lim_{\mu \to 0} \frac{\sin \mu}{\mu}}^{1}$

$\frac{1}{3}\mu := r^2$

$= 3$.

(8) Use the linear approximation to $f(x,y) = x^2/y$ at $(1,1)$ to approximate $0.99^2/1.01$.

(A) 1
(B) 0.99
(C) 0.98
(D) 0.97
(E) 0.96
(F) 0.95
(G) none of the above

$\frac{\partial f}{\partial x} = \frac{2x}{y}$, $\frac{\partial f}{\partial y} = -\frac{x^2}{y^2}$

$f_x(1,1) = 2$, $f_y(1,1) = -1$, $f(1,1) = 1$

$L(x,y) = 1 + 2(x-1) - 1(y-1)$

$L(0.99, 1.01) = 1 + 2(-0.01) - 1(0.01) = 0.97$

(9) Which is an equation of the plane tangent to the surface $z + xe^{-2y} = y^2 e^z + 2$ at $(1, -1, 2)$?

(A) $e^2 x + (1 - e^2) z = 2 - e^2$
(B) $e^2 x + e^2 y + (1 - e^2) z = 2 - 2e^2$
(C) $e^2 y + (1 - e^2) z = 2 - 3e^2$
(D) $-e^2 x + (1 - e^2) z = 2 - 3e^2$
(E) $e^2 x + (1 - e^2) z = e^2 - 2$
(F) $e^2 x + e^2 y - e^2 z = -2e^2$
(G) none of the above

write $f(x,y,z) = z + xe^{-2y} - y^2 e^z$

$\nabla f = \langle e^{-2y}, -2xe^{-2y} - 2ye^z, 1 - y^2 e^z \rangle$

$\nabla f(1, -1, 2) = \langle e^2, 0, 1 - e^2 \rangle$

$e^2(x-1) + (1-e^2)(z-2) = 0$

$e^2 x + (1-e^2) z = 2 - e^2$

(10) With $f(x,y) = y^2 \ln x$, find the slope of the graph of $z = f(x,y)$ in the direction $\vec{v} = \langle -1, 1 \rangle$, "over" the point $p = (1,2)$.

(A) -4
(B) -2
(C) 0
(D) 2
(E) 4
(F) none of the above

$\nabla f = \langle y^2/x, 2y \ln x \rangle$

$\nabla f(1,2) = \langle 4, 0 \rangle$

$(D_{\frac{1}{\sqrt{2}}\langle -1,1 \rangle} f)(1,2) = \langle 4, 0 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = \boxed{-2\sqrt{2}}$

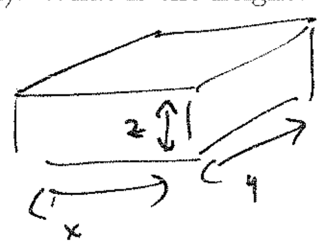
(11) With $f(x,y)$ as in problem (10), the direction of steepest slope of the graph at $(1,2)$ makes what ~~(counterclockwise)~~ angle with the vector $\langle 1, 0 \rangle$?

(A) 0°
(B) 45°
(C) 90°
(D) 135°
(E) 180°
(F) none of the above

is $\nabla f|_{(1,2)} = \langle 4, 0 \rangle$

(12) Determine the dimensions of a rectangular box without a lid, of volume 4 m^3, which minimize the surface area (hence material used). What is the height?

(A) $\frac{1}{\sqrt[3]{4}}$
(B) $\frac{1}{\sqrt[3]{2}}$
(C) 1
(D) $\sqrt[3]{2}$
(E) $\sqrt[3]{4}$
(F) 2
(G) 4
(H) none of the above



$V = xyz = 4 \text{ m}^3 \rightsquigarrow z = \frac{4}{xy}$

$S = xy + 2yz + 2xz$

$= xy + 8x^{-1} + 8y^{-1}$

$0 = \frac{\partial S}{\partial x} = y - \frac{8}{x^2}$ $\Big]$ $y = 8/x^2$

$0 = \frac{\partial S}{\partial y} = x - \frac{8}{y^2} = x - \frac{x^4}{8} \rightsquigarrow x = \cancel{0} \text{ or } 2$

So $y = 2$ also,

and then $z = 1$.

MATH 233 MIDTERM EXAM 2

PART II: HAND-GRADED PROBLEMS

This part has two pages. Show all the work you want graded for each problem in the space provided. Please **print your name at the top of each page.**

(1) [15 points] Find all critical points of $f(x, y) = x^3 + y^3 - 6xy$ on the xy-plane. Use the second-partials test to decide whether each point gives a local maximum, minimum, or saddle point.

$$0 = \frac{\partial f}{\partial x} = 3x^2 - 6y \quad , \quad 0 = \frac{\partial f}{\partial y} = 3y^2 - 6x$$

$$x^2 = 2y \qquad\qquad y^2 = 2x$$

$$\frac{x^2}{2} = y$$

$$2x = \left(\frac{x^2}{2}\right)^2 = \frac{x^4}{4}$$

$$8x = x^4 \implies x = 0\ (y=0) \text{ or } x = 2\ (y = 2).$$

$$\frac{\partial^2 f}{\partial x \partial y} = -6 \quad , \quad \frac{\partial^2 f}{\partial x^2} = 6x \quad , \quad \frac{\partial^2 f}{\partial y^2} = 6y .$$

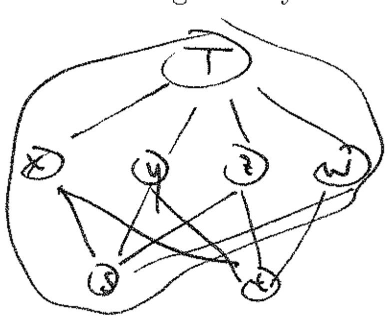
So matrix of 2nd partials is $\begin{bmatrix} 6x & -6 \\ -6 & 6y \end{bmatrix}$ with determinant

$$D = 36(xy - 1).$$

$\underline{(0,0)}$: $D < 0 \implies$ $\underline{\text{saddle}}$

$\underline{(2,2)}$: $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0 \implies$ $\underline{\text{local minimum}}$

(2) [5 points] If $T = f(x, y, z, w)$, and x, y, z, w are each functions of s and t, write a chain rule for $\partial T/\partial s$. Draw a diagram if you like.



$$\rightsquigarrow \frac{\partial T}{\partial s} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial T}{\partial z}\frac{\partial z}{\partial s} + \frac{\partial T}{\partial w}\frac{\partial w}{\partial s} .$$

(3) [10 points] Compute (or show does not exist): $\lim_{(x,y)\to(0,0)} \frac{xy+y^3}{x^2+y^2}$.

Consider the two paths ① $t \mapsto (t,0)$ and ② $t \mapsto (t,t)$.

We compute the limit as $t \to 0$ along each:

① $\lim_{t\to 0} \frac{t\cdot 0+0^3}{t^2+0^2} = \lim_{t\to 0} 0 = 0$

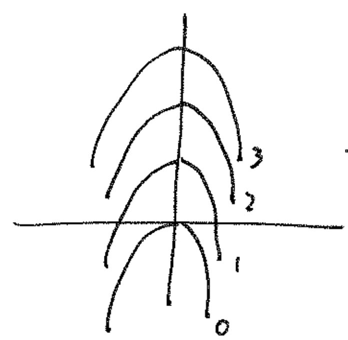
② $\lim_{t\to 0} \frac{t\cdot t+t^3}{t^2+t^2} = \lim_{t\to 0} \frac{t^2+t^3}{2t^2} = \lim_{t\to 0} \frac{1+t}{2} = \frac{1}{2}$.

$\neq$

$\Rightarrow$ DNE.

(4) [6 points] Sketch ~~(a) the graph of $f(x,y) = e^{-(x^2+y^2)}$~~; REMOVE (a) a rough contour map (level curves) for $f(x,y) = x^2+y$, and (b) the largest set on which $f(x,y) = \ln(1-x^2-y^2)$ is continuous.

(a)

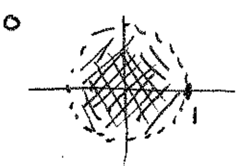


$k = x^2+y$

$\to y = k - x^2$

(b) $1-x^2-y^2>0$

$x^2+y^2<1$



(5) [4 points] I want a function $f(x,y)$ with $f_x(x,y) = \cos(x) - \sin(y)$ and $f_y(x,y) = \cos(y) + \sin(x)$, but I'm having trouble finding one. What gives?

Impossible: you would have

$f_{xy} = -\cos y \neq \cos x = f_{yx}$ (which are continuous!),

violating Clairaut's theorem.