This exam consists of 20 multiple choice (machine-graded) problems, worth 5 points each (for a total of 100 points). No 3x5 cards or calculators are allowed. You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your ID number (not your SS number) on the top of your answer card, using one blank for each digit, and shade in the corresponding boxes. Also print your name at the top of your card.

(1) If \( F(x, y) = \ln(x^2 + xy + y^2) \), what is \( F_x(2, -1) \)?
   (A) 6
   (B) 5
   (C) 4
   (D) 3
   (E) 2
   (F) 1
   \[ F_x = \frac{2x + y}{x^2 + xy + y^2} \]
   \[ F_x(2, -1) = \frac{4 - 1}{4 - 2 + 1} = \frac{3}{3} = 1 \]

(2) The level curves of \( f(x, y) = \frac{e^{x^2}}{e^{y^2}} \) are ...
   (A) bell curves
   (B) ellipses
   (C) hyperbolas
   (D) parabolas
   (E) like the graph of \( \ln \)
   (F) lines

\[ \nabla f = \frac{e^{x^2}}{e^{y^2}} \cdot (2x, -2y) \]

(3) Find \( \vec{r}(\frac{\pi}{2}) \) if \( \vec{a}(t) = (-\cos t, \sin t) \), \( \vec{v}(0) = (1, 0) \), and \( \vec{r}(0) = (1, 3) \).
   (A) \( (-\frac{\pi}{2}, 2 - \frac{\pi}{2}) \)
   (B) \( (\frac{\pi}{2} + \pi) \)
   (C) \( (\frac{\pi}{2}, 2 + \frac{\pi}{2}) \)
   (D) \( (\frac{\pi}{2}, -\frac{\pi}{2}) \)
   (E) \( (-\frac{\pi}{2}, 2 + \frac{\pi}{2}) \)
   (F) \( (-\pi, 2 + \pi) \)

\[ \vec{v} = \int \vec{a} \, dt = (-\sin t, -\cos t) + \vec{C} \]
\[ \langle 1, 0 \rangle = \vec{v}(0) = \langle 1, -1 \rangle + \vec{C} \implies \vec{C} = \langle 1, 1 \rangle \]
\[ \vec{r} = \int \vec{v} \, dt = \langle t + \cos t, t - \sin t \rangle + \vec{K} \]
\[ \langle 1, 3 \rangle = \vec{r}(0) = \langle 1, 0 \rangle + \vec{K} \implies \vec{K} = \langle 0, 3 \rangle \]
So \( \vec{r}(\frac{\pi}{2}) = \langle \frac{\pi}{2}, 2 + \frac{\pi}{2} \rangle \)
and \( \vec{r}(\frac{\pi}{2}) = \langle \frac{\pi}{2}, 2 + \frac{\pi}{2} \rangle \)
(4) After you are picked up by your Uber driver in St. Louis, your path of motion is described by \( \mathbf{r}(t) = (2 \cos t, 2 \sin t, \sqrt{5}t) \) (in meters) from \( t = 0 \) to \( t = 10 \) (seconds). Calculate the length (in meters) of the arc along which you traveled (in a parking lot or tornado, as the case may be).

(A) 15
(B) 30
(C) 45
(D) 60
(E) 75
(F) 90

\[ \mathbf{r}'(t) = \langle 2 \cos t, -2 \sin t, \sqrt{5} \rangle \]
\[ \| \mathbf{r}'(t) \| = \sqrt{4 \cos^2 t + 4 \sin^2 t + 5} = \sqrt{9} = 3 \]

Arc length: \[ \int_0^{10} \| \mathbf{r}'(t) \| \, dt = \int_0^{10} 3 \, dt = 30 \]

(5) If the temperature distribution \( T(x, y, z) \) has \( T_x(0, 2, 0) = 1, T_y(0, 2, 0) = -1, \) and \( T_z(0, 2, 0) = -\sqrt{5} \) (in degrees Celsius), how fast (in degrees/second) is the outside temperature changing along your path in problem (4) at time \( t = 0 \)?

(A) -3
(B) -2
(C) -1
(D) 0
(E) 1
(F) 2

Use Chain rule:

First, \( \mathbf{r}'(0) = \langle 0, 2, 0 \rangle \) and

\[ x(t) = 2 \sin t, \quad y(t) = 2 \cos t, \quad z(t) = \sqrt{5} t \]

\[ x'(t) = 2 \cos t, \quad y'(t) = -2 \sin t, \quad z'(t) = \sqrt{5} \]

\[ \left. \frac{dT}{dt} \right|_{t=0} = T_x(0, 2, 0) x'(0) + T_y(0, 2, 0) y'(0) + T_z(0, 2, 0) z'(0) \]

\[ = 1 \cdot 2 + (-1) \cdot 0 + (-\sqrt{5}) \cdot \sqrt{5} \]

\[ = -5 \]

(6) Which of the following could be \( f_x(x, y) \) if \( f_y(x, y) = 2x^3y - y^3 \)?

(A) \( 2 - y^3 + 3x^2y^2 \)
(B) \( \frac{1}{2}x^4 - 3y^2 \)
(C) \( x^2y^2 - 3y^2 + 1 \)
(D) \( e^x - 6 + x^2y^2 \)
(E) \( 5 + x^3 + 3x^2y^2 \)
(F) \( x^4 + 2 \)

Use Chain rule:

\( f_x = \frac{\partial}{\partial x} (f_y) \cdot (2x^3y - y^3) = 6x^2y \)

Taking antiderivative with respect to \( y \), we get

\( f_x = 3x^2y^2 + (\text{any function of } x) \)
(7) Suppose a tree trunk has a radius of 10 inches, currently increasing at \( \frac{1}{2} \) inch per year, and a height of 200 inches, currently increasing at 4 inches per year. (Assume the tree trunk is a right circular cylinder.) How fast is the volume of the tree trunk currently increasing, in cubic inches per year?

\[
V = \pi r^2 h
\]

Use Chain rule:

\[
\frac{dV}{dt} = V_r \cdot r'(t) + V_h \cdot h'(t)
\]

\[
= \left(2\pi r h\right) \cdot r'(t) + \left(\pi r^2\right) \cdot h'(t)
\]

\[
= 2\pi(10)(200) \cdot \frac{1}{2} + \pi(10)^2 \cdot 4
\]

\[
\frac{dV}{dt} = 2400\pi
\]

(8) Which of the following limits exist: (a) \( \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \); (b) \( \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2 + y^2} \); (c) \( \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} \)?

(A) (a) only
(B) (b) only
(C) (c) only
(D) (a) and (c)
(E) (a) and (b)
(F) (b) and (c)

(a) DNE: e.g. consider \( x = 0 \) gives \( \frac{y}{y^2} = y \to 0 \)

(b) DNE: e.g. use polar form \( x = r \cos \theta, y = r \sin \theta \)

(c) is 0: use 0 \( \leq \frac{y^2}{x^2 + y^2} \leq 1 \)

\[
-|x| \leq \frac{xy}{x^2 + y^2} \leq |x|
\]

(Squeeze lemma)

(9) Find the equation for the tangent plane to \( z = xe^{-2y} \) at \( (1,0,1) \).

(A) \( 1 = x - 2y + z \)
(B) \( z = x \)
(C) \( z = x + 2y \)
(D) \( 1 = x + 2y + z \)
(E) \( z = x - 2 \)
(F) \( z = x - 2y \)

Let \( f(x,y) = xe^{-2y} \). Use linear approximation:

\[
f_x = e^{-2y}, \quad f_y = -2xe^{-2y}
\]

\[
f_x(1,0) = 1, \quad f_y(1,0) = -2
\]

\[
z = L(x,y) = f_x(1,0)(x-1) + f_y(1,0)(y-0) + f(1,0)
\]

\[
z = x - 1 + (-2)y + 1
\]

Tangent plane

\[
z = x - 2y
\]
(10) Find the minimum distance between the point \( (1, 2, 0) \) and the quadric cone \( z^2 = x^2 + 7y^2 \).

\[
\text{(Distance)}^2 = (x-1)^2 + (y-2)^2 + z^2
\]
\[
= (x-1)^2 + (y-2)^2 + x^2 + 7y^2
\]
\[
= 2x^2 + 8y^2 - 2x - 4y + 5 = F(x, y)
\]

Set \( \begin{cases} 0 = F_x = 4x - 2 \Rightarrow x = \frac{1}{2} \\ 0 = F_y = 16y - 4 \Rightarrow y = \frac{1}{4} \end{cases} \) \( \Rightarrow \) distance \( = \sqrt{F(\frac{1}{2}, \frac{1}{4})} = \sqrt{4} = 2 \).

(11) With \( f(x, y) = x^2y \), find the slope of the graph of \( z = f(x, y) \) in the direction \( \vec{v} = (3, -4) \), “over” the point \( P(1, 2) \).

\[ \text{Directional Derivative:} \quad \hat{u} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 3, -4 \rangle}{\sqrt{3^2 + 4^2}} = \langle \frac{3}{5}, -\frac{4}{5} \rangle \]

\[ \nabla f = \langle 2xy, x^2 \rangle \Rightarrow \nabla f(1, 2) = \langle 4, 1 \rangle \]

\[ (\hat{u}, f)(1, 2) = (\nabla f)(1, 2) \cdot \langle \frac{3}{5}, -\frac{4}{5} \rangle = \frac{8}{5} \]

(12) With \( f \) and \( P \) as in problem (11), which vector is tangent to the level curve of \( f(x, y) \) through \( P \)?

\[ \langle \nabla f \rangle (1, 2) \quad \text{is} \quad \perp \quad \text{to the level curve} \]

\[ \text{so you need something} \quad \perp \quad \text{to} \quad \langle 4, 1 \rangle \]

\[ \boxed{\langle 1, -4 \rangle} \]
(13) Which of the following is the (absolute) maximum value of \( f(x,y) = x^2 - y^2 + 1 \) on the set \( D = \{(x,y) | -1 \leq x \leq 3, -1 \leq y \leq 4\} \)?

(A) 1
(B) 5
(C) 9
(D) 10
(E) 14
(F) 26

Boundary:
- \( f(-1, y) = 2 - y^2 \) max = 2
- \( f(3, y) = 10 - y^2 \) max = 9
- \( f(x, -1) = x^2 \) max = -6
- \( f(x, 4) = x^2 - 15 \) both at \( x = 3 \)

Interior:
- \( O = (x, y) \Rightarrow (x, y) = (0, 0) \)
- \( f(0, 0) = 1 \)

Since the maximum must be among these values, it is 10.

(14) The radius of the osculating circle to \( \vec{r}(t) = (2t, \cos t, \sin t) \) is everywhere equal to:

(A) \( \frac{1}{5} \)
(B) 1
(C) 5
(D) \( \frac{2}{\sqrt{5}} \)
(E) \( \frac{4}{\sqrt{5}} \)
(F) \( \frac{6}{\sqrt{5}} \)

\[ \vec{r}' = \langle 2, -\sin t, \cos t \rangle \Rightarrow ||\vec{r}'|| = \sqrt{5} \]
\[ \vec{r}'' = \langle 0, -\cos t, -\sin t \rangle \]
\[ \vec{r}' \times \vec{r}'' = \langle 1, 2\sin t, -2\cos t \rangle \Rightarrow ||\vec{r}' \times \vec{r}''|| = \sqrt{5} \]
\[ R = \frac{||\vec{r}'||}{||\vec{r}''||^2} = \frac{\sqrt{5}}{(\sqrt{5})^2} = \frac{1}{5} \Rightarrow R = 5 \]

(15) The osculating plane (for \( \vec{r} \) as in problem (14)) at \( t = \frac{\pi}{2} \) has equation:

(A) \( x + 2z = \pi + 2 \)
(B) \( x - 2y = \pi \)
(C) \( x + 2y = \pi + 2 \)
(D) \( x - 2y = \pi - 2 \)
(E) \( x + 2y = \pi \)
(F) \( x - 2y = \pi - 2 \)

At \( t = \frac{\pi}{2} \), get
\[ \vec{r}(\frac{\pi}{2}) = \langle \pi, 0, 1 \rangle \leftarrow \text{plane contains this point} \]
\[ \vec{B} \text{ is a multiple of } \vec{r}' \times \vec{r}'' \text{, normal to osculating plane.} \]

\[ \vec{B}(\frac{\pi}{2}) = \text{mult. of } \langle 1, 2, 0 \rangle \]
\[ \implies \hat{O} = 1 \langle x - \pi \rangle + 2 \langle y - 0 \rangle + 0 \langle z - 1 \rangle = x + 2y - \pi \]
\[ \implies \text{equ. is } \hat{u} = x + 2y \]
(16) How many of each type of critical point does \( f(x, y) = 2x^3 - x^2 + 3y^2 \) have: saddle point; local maximum; local minimum?

(A) 1; 2; 0
(B) 1; 1; 1
(C) 0; 0; 2
(D) 1; 0; 2
(E) 0; 2; 0
(F) 0; 1; 1

\[
D = \begin{pmatrix}
  f_{xx} & f_{xy} \\
  f_{xy} & f_{yy}
\end{pmatrix} = \begin{pmatrix}
  24x^2 - 2 & 0 \\
  0 & 6
\end{pmatrix} = 12(12x^2 - 1)
\]

\[
D(0, 0) = -12 < 0 \implies \text{Saddle}
\]

\[
D\left(\frac{1}{2}, 0\right) = D\left(-\frac{1}{2}, 0\right) = 24 > 0 \implies 2 \text{ local minima}
\]

(17) The formula \( \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \) determines the combined resistance \( R \) when resistors of resistance \( R_1 \) and \( R_2 \) are connected in parallel. Suppose that \( R_1 \) and \( R_2 \) were measured at 25 and 100 ohms, respectively, so that you calculate \( R = 20 \). If the possible errors in each of your two measurements were ±0.5 ohms each, use differentials to estimate the maximum possible error (in ohms) in the computed value of \( R \).

(A) 0.16
(B) 0.22
(C) 0.28
(D) 0.34
(E) 0.40
(F) 0.46

\[
-\frac{1}{R^2} \, dR = -\frac{1}{R_1^2} \, dR_1 - \frac{1}{R_2^2} \, dR_2
\]

\[
\frac{dR}{R} = \frac{R_1}{R_1^2} \, dR_1 + \frac{R_2}{R_2^2} \, dR_2 = \left(\frac{25}{25^2}\right) \cdot 0.5 + \left(\frac{100}{100^2}\right) \cdot 0.5
\]

\[
= \frac{1}{50} \cdot (0.5) = \frac{17}{50} = 0.34
\]

(18) What is the maximum curvature of the curve \( y = \sqrt{3} \ln(x) \)?

(A) \( \frac{1}{3} \)
(B) \( \frac{1}{5} \)
(C) \( \frac{1}{9} \)
(D) \( \frac{1}{3} \)
(E) \( \frac{1}{5} \)
(F) \( \frac{1}{9} \)

\[
\vec{r}(t) = \langle \sqrt{3}, \sqrt{3} \ln t, 0 \rangle
\]

\[
\vec{r}'(t) = \langle 1, \sqrt{3}, 0 \rangle \implies \|\vec{r}'\| = \sqrt{1 + \frac{3}{t^2}}
\]

\[
\vec{r}''(t) = \langle 0, -\frac{\sqrt{3}}{t}, 0 \rangle \implies \vec{r}' \times \vec{r}'' = \langle 0, 0, -\sqrt{3} \rangle
\]

\[
\implies \kappa = \frac{\sqrt{3}}{t^2 + 1} = \frac{\sqrt{3}}{t^2 + (t^2 + 3)\sqrt{3}}
\]

\[
= \sqrt{3} \frac{t^2 + 3 - \sqrt{3}}{t^2 + (t^2 + 3)\sqrt{3}} = \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2} - t}{\sqrt{2} + t} = \frac{\sqrt{2}}{2} \cdot \frac{2\sqrt{2}}{2t} = \frac{2}{\sqrt{2}t}
\]

and \( \kappa(\sqrt{2}) = \frac{3/(\sqrt{2})}{(3\sqrt{2} + 2)\sqrt{2}} = \frac{2}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{2} = \frac{2}{\sqrt{2t}} = \frac{2}{\sqrt{2t}} \)}
(19) If a batted baseball leaves the bat at an angle of 45° with the horizontal, and is caught by a Cardinals outfielder 200 feet from home plate, what was the original velocity (in feet/second) of the ball? [Use \( g = 32 \text{ ft/sec}^2 \)]

(A) 55  
(B) 60  
(C) 65  
(D) 70  
(E) 75  
(F) 80

\[
\vec{v}_0 = \left( \frac{v_0}{\sqrt{2}}, \frac{v_0}{\sqrt{2}} \right) \\
\vec{a} = \left( 0, -32 \right) \implies \vec{v} = \left( \frac{v_0}{\sqrt{2}}, \frac{v_0}{\sqrt{2}} - 32t \right) \\
\implies \vec{r} = \left( \frac{v_0 t}{\sqrt{2}}, \frac{v_0 t}{\sqrt{2}} - 16t^2 \right)
\]

Set \( \left( 200, 0 \right) = \vec{r}(t_0) = \left( \frac{v_0 t_0}{\sqrt{2}}, \frac{v_0 t_0}{\sqrt{2}} - 16t_0^2 \right) \)

\[
\begin{align*}
\frac{v_0 t_0}{\sqrt{2}} &= 16t_0^2 \implies t_0 = \frac{v_0}{16\sqrt{2}} \\
\frac{v_0 t_0}{\sqrt{2}} &= 200 \implies \frac{v_0}{\sqrt{2}} \cdot \frac{v_0}{16\sqrt{2}} = 200 \implies v_0 = 400 \cdot 16 \\
\implies v_0 &= 80 \cdot 4 = 320
\end{align*}
\]

(20) Paris is located at the origin of the \( xy \)-plane. Rail lines emanate from Paris along all rays, and these are the only rail lines. (Yes, there are infinitely many.) Let \( f(x, y) \) be the distance from \((x, y)\) to \((1, 0)\) on the French railroad. Determine the set of points at which \( f \) is discontinuous.

(A) positive \( x \)-axis \((x > 0, y = 0)\)  
(B) nonnegative \( x \)-axis \((x \geq 0, y = 0)\)  
(C) entire \( x \)-axis \((y = 0)\)  
(D) negative \( x \)-axis \((x < 0, y = 0)\)  
(E) the origin \((0, 0)\) only  
(F) entire \( xy \)-plane

\[
\text{of } f(x, y) \\
\begin{align*}
\text{Limit as } (x, y) \to (a, 0) \text{ from above or below is } a+1 \\
\text{Limit as } (x, y) \to (a, 0) \text{ along the } (+)-x\text{-axis is } |a-1|. \\
\text{This is also } f(a, 0).
\end{align*}
\]

So for \( a > 0 \), we do \textbf{NOT} have \( \lim_{(x, y) \to (a, 0)} f(x, y) = f(a, 0) \)  
(i.e., the limit doesn’t even exist!).