This exam consists of 25 multiple choice (machine-graded) problems, worth 4 points each. No 3x5 cards or calculators are allowed. You will need a pencil to mark your card. If you do not have one, please ask your proctor. Write your ID number (not your SS number) on the six blank lines on the top of your answer card, using one blank for each digit, and shade in the corresponding boxes. Also print your name at the top of your card. Good luck!

Some possibly useful formulas: $\ln(x) - \ln(y) = \ln(x/y)$, $\sin^2 \eta = 1 - \cos^2 \eta = \frac{1}{2}(1 - \cos(2\eta))$.

1) Find the area of the triangle in space with coordinates $(1, 2, -3)$, $(5, -1, 1)$, and $(1, 3, -2)$.

\[ \frac{1}{2} \| \langle 4, -3, 4 \rangle \times \langle 0, 1, 1 \rangle \| = \frac{1}{2} \| \langle -7, -4, 4 \rangle \| \]
\[ = \frac{1}{2} \sqrt{49 + 16 + 16} = \frac{1}{2} \sqrt{81} = \frac{9}{2} \]

2) Let $P$ be the plane which contains the triangle of problem (1), and $Q$ the plane with equation $x - 2y + 2z = 6$. What is the cosine of the angle between $P$ and $Q$?

\[ \cos \theta = \frac{\langle -7, -4, 4 \rangle \cdot \langle 1, -2, 2 \rangle}{\| \langle -7, -4, 4 \rangle \| \| \langle 1, -2, 2 \rangle \|} = \frac{9}{9 \cdot 3} = \frac{1}{3} \]

3) Evaluate $f_{xy}(1, 0)$ for $f(x, y) = \sin(x^2y)$.

\[ f_x = 2xy \cos(x^2y) \]
\[ (f_x)_y = 2x \cos(x^2y) - 2x^3y \sin(x^2y) \]
\[ f_{xy}(1, 0) = 2 \cdot 1 \cdot \cos 0 = 2 \]

1
(4) Determine the steepest (i.e. maximum) slope of \( z = f(x, y) = x^3 + xy + 4y \) at \((x, y) = (2, -4)\).

(A) 5
(B) 6
(C) 7
(D) 8
(E) 9
(F) 10

\[ \nabla f = \langle 3x^2 + y, x + 4 \rangle \]

\[ \| \nabla f(2, -4) \| = \| \langle 8, 6 \rangle \| = \sqrt{64 + 36} = 10 \]

(5) Evaluate \( \iint_D \frac{1}{\sqrt{x+y^2}} \, dA \) where \( D \) is the region between the circles \( x^2 + y^2 = 4 \) and \( x^2 + y^2 = 9 \).

(A) \( \pi \ln(2) \)
(B) \( \frac{\pi}{3} \)
(C) \( \pi \)
(D) \( \pi \ln\left(\frac{3}{2}\right) \)
(E) \( \frac{4\pi}{3} \ln\left(\frac{3}{2}\right) \)
(F) \( \frac{5}{2} \)

\[ \int_0^{2\pi} \int_2^3 \frac{1}{r} \, r \, dr \, d\theta = 2\pi \int_2^3 \frac{1}{r} \, dr \]

\[ = 2\pi \left( \ln(3) - \ln(2) \right) \]

\[ = 2\pi \ln\left(\frac{3}{2}\right) \]

(6) Calculate the iterated integral \( \int_0^1 \int_{x^2}^{\sqrt{x}} 4xy \, dy \, dx \).

(A) 0
(B) \( \frac{1}{3} \)
(C) \( \frac{1}{4} \)
(D) \( \frac{1}{3} \)
(E) \( \frac{1}{2} \)
(F) 1

\[ = \int_0^1 \left[ 2xy^2 \right]_{y=x^2}^{y=\sqrt{x}} \, dx \]

\[ = \int_0^1 \left( 2x^3 - 2x^3 \right) \, dx \]

\[ = \left[ \frac{2x^3}{3} - \frac{x^4}{2} \right]_0^1 \]

\[ = \frac{2}{3} - \frac{1}{2} \]

\[ = \frac{1}{6} \]
(7) Suppose we are integrating in \(x\) and \(y\) and we want to integrate in \(u\) and \(v\), where \(x = \frac{u^2}{v}\) and \(y = \frac{v^3}{u}\). We must replace \(d\!x\,dy\) by what function times \(du\,dv\)?

(A) \(7v^2 + 5u^2\)
(B) \(9u^2\)
(C) \(7v^2\)
(D) \(4v^2 + 7u^2\)
(E) \(5u^2\)
(F) \(9v^2\)

\[
\begin{vmatrix}
x_u & x_v \\
y_u & y_v
\end{vmatrix} = \begin{vmatrix}
2v/\sqrt{v} & -u^2/\sqrt{v} \\
-v^4/\sqrt{u} & 4v^3/\sqrt{u}
\end{vmatrix} = \frac{2u}{\sqrt{v}} \frac{4v^3}{\sqrt{u}} - \frac{u^2}{\sqrt{v^3} \sqrt{u}}
\]

\[
= 8v^2 - v^4
\]

\[
= 7v^2
\]

(8) For which of the vector fields

(i) \(yi + xj\), \hspace{1cm} (ii) \(-yi + xj\), \hspace{1cm} (iii) \(\frac{-y}{x^2 + y^2}i + \frac{x}{x^2 + y^2}j\)

is the line integral zero along any closed path encircling the origin?

(A) (i) only
(B) (ii) only
(C) (iii) only
(D) (i) and (ii)
(E) (ii) and (iii)
(F) (i) and (iii)

Check \(\text{curl} = \frac{Q_x - P_y}{j}\):

0 for (i) \& (iii)

but (iii) has a singularity (and is our example from class). So only (i) is conservative.

(9) Let \(f\) be a function (scalar field) and \(\vec{F}\) a vector field, both on \(\mathbb{R}^3\). How many of \(\nabla f\), \(\text{curl}(\vec{F})\), \(\text{div}(\vec{F})\), \(\text{div} (\text{div}(\vec{F}))\), \(\text{curl} (\text{curl}(\vec{F}))\), \(\text{curl} (\text{div}(\vec{F}))\) are: scalar fields, vector fields, meaningless?

(A) 3;2;1
(B) 2;1;3
(C) 1;3;2
(D) 2;3;1
(E) 3;1;2
(F) 1;2;3
(10) How many of the vector fields

\[ 3y^2zi + (y - x^3)k, \quad 3y^2zi + xyj - x^3k, \quad 3y^2zi + x^3j + yzk, \]

\[ (3y^2z - xy)i + x^3j + yzk, \quad -xyi + \frac{1}{2}y^2j + e^{x^2y^2}k \]

on \( \mathbb{R}^3 \) are the curl of another vector field?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

\[ \check{\nabla} \equiv \begin{array}{ccc} 0 & 1 & y \\ x & 0 & 0 \\ 0 & 0 & 0 \end{array} \]

\[ \text{curl} = 0 \]

(11) Determine the value of \( a \) (if any) that makes \( \vec{F}(x, y, z) = y^2i + (axy + e^{z^2})j + 3(ayz)e^{z^2}k \) a conservative field on \( \mathbb{R}^3 \).

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

\[ \begin{align*}
\text{need:} & \quad P_y = Q_x \\
& \quad 2y = ay \\
\text{need:} & \quad P_z = Q_x \\
& \quad 0 = 0 \\
\text{need:} & \quad Q_x = R_y \\
& \quad 6\pi e^{3e^2} = 3ae^{3+e} \\
\text{so} & \quad a = 2
\end{align*} \]

(12) Describe the flow lines of \( \vec{F}(x, y) = xi - yj \).

(A) parabolas
(B) hyperbolas
(C) circles
(D) ellipses
(E) lines
(F) none of these

\[ \begin{align*}
\dot{x}(t) &= x(t) \implies x(t) = Ce^t \\
\dot{y}(t) &= -y(t) \implies y(t) = Ke^{-t} = \frac{C_0}{x(t)} \\
y &= \frac{C_0}{x}
\end{align*} \]
(13) Evaluate the line integral \( \int_C xe^{xy} \, ds \), where \( C \) is the line segment from \((0,0)\) to \((3,4)\).

\[
\begin{align*}
(A) & \quad \frac{1}{6} (e^3 - 1) \\
(B) & \quad \frac{1}{6} (e^4 - 1) \\
(C) & \quad \frac{1}{6} (e^{12} - 1) \\
(D) & \quad \frac{3}{2} (e^3 - 1) \\
(E) & \quad \frac{3}{8} (e^{12} - 1) \\
(F) & \quad \frac{3}{8} (e^{12} - 1)
\end{align*}
\]

\[
\begin{align*}
&= \int_0^1 3te^{(2t+2)} - 5 \, dt \\
&= 15 \int_0^1 t e^{12t^2} \, dt \\
&= \frac{15}{24} \left[ e^{12t^2} \right]_0^1 \\
&= \frac{5}{8} (e^{12} - 1).
\end{align*}
\]

(14) Find the work done by the force field \( \vec{F}(x, y) = x^3 i + 2x^2 y j \) in moving an object along the quarter-circle of radius 1 (centered at \((0,0)\)) from \((1,0)\) to \((0,1)\).

\[
\begin{align*}
(A) & \quad 0 \\
(B) & \quad \frac{1}{6} \\
(C) & \quad \frac{1}{2} \\
(D) & \quad \frac{1}{3} \\
(E) & \quad \frac{1}{8} \\
(F) & \quad 1
\end{align*}
\]

\[
\begin{align*}
W & = \int_C \vec{F} \cdot \, ds \\
& = \int_0^{\pi/2} \cos^3 t (-\sin t \, dt) + 2\cos^2 t \sin t \cos t \, dt \\
& = \int_0^{\pi/2} \cos^3 t \sin t \, dt \\
& = -\frac{1}{4} \left[ \cos^4 t \right]_0^{\pi/2} \\
& = \frac{1}{4}.
\end{align*}
\]

(15) Let \( C \) be an arbitrary (possibly curved) path from \((1,1)\) to \((2,3)\). Compute the line integral of \( \vec{F}(x, y) = (4x^3 - 2xy)i + (2 - x^2)j \) along \( C \).

\[
\begin{align*}
(A) & \quad 0 \\
(B) & \quad 2 \\
(C) & \quad 4 \\
(D) & \quad 6 \\
(E) & \quad 8 \\
(F) & \quad 10
\end{align*}
\]

\[
\begin{align*}
\vec{F} & = \nabla f, \quad \text{where} \\
f & = x^4 - x^2 y + 2y \\
&\int_C \nabla f \cdot \, ds \\
&= f(2,3) - f(1,1) \\
&= 2^4 - 2^2 \cdot 3 + 2 \cdot 3 - (1 - 1 + 2) \\
&= 8.
\end{align*}
\]
(16) What is the integral of the vector field \( \vec{F}(x, y) = (yx^2e^{x^2}+5y)i + (\frac{1}{3}e^{x^2}+8x)j \) over the (counterclockwise) circle \( C \) of radius 2 (centered about \((0,0))\)? [Hint: you won’t be able to compute this line integral directly. Use a theorem.]
(A) 2\(\pi\)  
(B) 4\(\pi\)  
(C) 6\(\pi\)  
(D) 8\(\pi\)  
(E) 10\(\pi\)  
(F) 12\(\pi\)

\[ \oint_C \vec{F} \cdot d\vec{r} = \iint_S (Q_x - P_y) \, dA \]
\[ = 3 \frac{\pi(8)}{\pi(2)} \]
\[ = 12\pi \]

(17) Find the area of the surface with parametric equations \( x = u^2, y = uv, z = \frac{1}{2}v^2 \), \( 0 \leq u \leq 1, 0 \leq v \leq 1. \)
(A) 0  
(B) 1  
(C) 2  
(D) 3  
(E) 4  
(F) 5

\[ \vec{P}_u \times \vec{P}_v = \langle 2u, v, 0 \rangle \times \langle 0, u, v \rangle \]
\[ = \langle v^2, -2uv, 2u^2 \rangle \]
\[ \|\vec{P}_u \times \vec{P}_v\| = \sqrt{v^4 + 4u^2v^2 + 4u^4} = \sqrt{(v^2 + 2u^2)^2} \]
\[ = v^2 + 2u^2 \]

\[ A = \iint_D (v^2 + 2u^2) \, du \, dv \]
\[ = \int_0^1 2u^2 \, du + \int_0^1 v^2 \, dv \]
\[ = \frac{2}{3}u^3 \bigg|_0^1 + \frac{v^3}{3} \bigg|_0^1 = \frac{2}{3} + \frac{1}{3} = 1 \]
(18) Calculate the iterated integral \( \int_0^{\sqrt{3}} \int_x^{\sqrt{x^2}} \cos(y^2) \, dy \, dx \). [Hint: reverse the order.]

(A) 0  
(B) \frac{1}{3}  
(C) \frac{1}{4}  
(D) \frac{1}{2}  
(E) \frac{5}{6}  
(F) 1

\[
\begin{align*}
\int_0^{\sqrt{3}} \int_x^{\sqrt{x^2}} \cos(y^2) \, dy \, dx &= \int_0^{\sqrt{3}} \left[ \int_0^y \cos(y^2) \, dy \right] \, dx \\
&= \int_0^{\sqrt{3}} y \cos(y^2) \, dy \\
&= \frac{1}{2} \left[ \sin(y^2) \right]_0^{\sqrt{3}} \\
&= \frac{1}{2} \sin \left( \frac{3}{2} \right) = \frac{1}{2}
\end{align*}
\]

(19) Describe the curve \( r = \tan \theta \sec \theta \).

(A) parabola  
(B) hyperbola  
(C) circle  
(D) ellipse  
(E) line  
(F) none of these

\[
r = \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta} = \frac{r \sin \theta}{r \cos \theta} \cdot \frac{r}{r \cos \theta}
\]

\[
x^2 = y
\]
(20) Determine the arclength of the curve traced out by \( \mathbf{r}(t) = \left( \frac{1}{3} t^3, t^2, 2t \right) \) for \( 0 \leq t \leq 3 \).

(A) 9  
(B) 12  
(C) 15  
(D) 18  
(E) 21  
(F) 24

\[ \left\| \mathbf{r}'(t) \right\| = \sqrt{t^4 + 4t^2 + 4} \]

\[ \int_0^3 \sqrt{t^4 + 4t^2 + 4} \, dt \]

\[ = \int_0^3 (t^2 + 2) \, dt \]

\[ = \left[ \frac{t^3}{3} + 2t \right]_0^3 \]

\[ = \frac{27}{3} + 2 \cdot 3 = 9 + 6 = 15 \]

(21) For \( \mathbf{r}(t) \) as in problem (20), compute the radius of the osculating circle at \( t = 0 \).

(A) 1  
(B) 2  
(C) 3  
(D) 4  
(E) 5  
(F) 6

\[ R(t) = \frac{1}{\kappa(t)} = \frac{\left\| \mathbf{r}'(t) \right\|^3}{\left\| \mathbf{r}'(t) \times \mathbf{r}''(t) \right\|} \]

\[ \mathbf{r}'(0) = \left< 0, 0, 2 \right> \]

\[ \mathbf{r}''(0) = \left< 2, 0, 0 \right> \]

\[ \mathbf{r}'(0) \times \mathbf{r}''(0) = \left< -4, 0, 0 \right> \]

\[ R(0) = \frac{2^3}{4} = 2 \]
(22) A triangle has vertices \( A, B, \) and \( C \). The length of the side \( AB \) is 10 inches, and is increasing at a rate of 3 inches/second. The length of the side \( AC \) is 8 inches, and is decreasing at a rate of 4 inches/second. The angle \( \theta \) between \( AB \) and \( AC \) is \( \frac{\pi}{6} \) and increasing at \( \frac{\sqrt{3}}{10} \) radians/second. How fast (in in\(^2/\)sec) is the area changing?

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

\[
\text{Area} = \frac{1}{2} ||\vec{AB} \times \vec{AC}|| = \frac{1}{2} ||\vec{AB}|| ||\vec{AC}|| \sin \theta = \frac{1}{2} xy \sin \theta
\]

\[x = 10, \ y = 8, \ \theta = \frac{\pi}{6}\]

\[x' = 3, \ y' = -4, \ \theta' = \frac{\sqrt{3}}{10}\]

\[A' = A_x x' + A_y y' + A_\theta \theta'\]

\[= \frac{1}{2}(y \sin \theta) x' + \frac{1}{2}(x \sin \theta) y' + \frac{1}{2}(xy \cos \theta) \theta'\]

\[= \frac{1}{2}(8 \cdot \frac{\sqrt{3}}{10}) 3 + \frac{1}{2}(10 \cdot \frac{\sqrt{3}}{6})(-4) + \frac{1}{2}(10 \cdot 8 \cdot \frac{\sqrt{3}}{10}) \frac{\sqrt{3}}{10}\]

\[= 6 - 10 + 6\]

\[= 2\]

(23) How many of each kind of critical point does \( f(x, y) = xy^2 - 6x^2 - 3y^2 \) have: local maximum; local minimum; saddle point?

(A) 1;0;1
(B) 0;1;1
(C) 2;0;1
(D) 1;1;1
(E) 1;0;2
(F) 0;1;2

\[0 = f_x = y^2 - 12x\]

\[0 = f_y = 2xy - 6y = 2y(x - 3)\]

\[D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -12 & 2y \\ 2x & -6 \end{vmatrix} = -4y^2 - 24x + 72\]

\[D(0,0) = 72 > 0, \ f_{xx}(0,0) = -12 < 0 \quad \text{no local max}\]

\[D(3,\pm 6) = -72 \pm 72 - 4 \cdot 3^2 < 0 \quad \text{no saddle point}\]
(24) Find the maximum of \( f(x, y) = xy \), subject to the constraint \( g(x, y) = 4x^2 + 9y^2 - 36 = 0 \).

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4
(F) 5

\[
\nabla f = \lambda \nabla g
\]

\[
\langle y, x \rangle = \lambda \langle 8x, 18y \rangle
\]

\( y = 8x \), \( x = 18 \lambda y = 18 \lambda \cdot 8 \cdot x = 9.16 \lambda^2 x \)

\[\Rightarrow x = 0 \quad \text{or} \quad \lambda = \pm \frac{1}{12} .\]

impossible, since it gives \( y = 0 \) and \( (0,0) \) doesn't satisfy constraint.

plug \( y = \pm \frac{\lambda}{12} \), \( x = \pm \frac{2}{3}x \) into constraint:

\[36 = 4x^2 + 9 \cdot \frac{4}{9} x^2 = 8x^2 \quad \Rightarrow \quad x = \pm \frac{36}{2} , \quad y = \pm \sqrt{2}
\]

\[\Rightarrow \quad y = \pm \frac{\sqrt{2}}{2} \cdot \sqrt{2} = 3 .\]

(25) The folium of Descartes is the curve \( x^3 + y^3 = 3xy \) in the plane.

The portion in the first quadrant may be parametrized by \( \vec{r}(t) = \frac{3t}{1+t^3} \mathbf{i} + \frac{3t^2}{1+t^3} \mathbf{j} \), as \( t \) runs from 0 to \(+\infty\). Use Green's theorem to find the area it encloses.

(A) \( \frac{1}{2} \)
(B) 1
(C) 2
(D) \( \frac{5}{2} \)
(E) \( \frac{5}{2} \)
(F) 3

\[
A(\vec{r}) = \frac{1}{2} \oint_C -y \, dx + x \, dy
\]

\[
y'(t) = \frac{3t (2-t^3)}{(1+t^3)^2}
\]

\[\Rightarrow \quad x'(t) = \frac{3(1-2t^3)}{(1+t^3)^2}
\]

\[
= \frac{3}{2} \left[ \frac{-1}{(1+t^3)} \right]_0^\infty = \frac{3}{2} .
\]