MATH 233 LECTURE 13 (§14.2): LIMITS OF MULTIVARIABLE FUNCTIONS

Limits.

- Let $f(x, y)$ be a function of two variables. We say that

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

  if for any given $\epsilon > 0$ (no matter how small), there exists a $\delta > 0$ (sufficiently small, and possibly depending on ϵ) so that whenever $0 < d((x, y), (a, b)) \leq \delta$, we have $|f(x, y) - L| \leq \epsilon$.
- The usual limit laws hold for sums, products and compositions of functions. Quotients of functions, on the other hand, can be quite problematic when the denominator is zero at (a, b).
- A clever trick that works on some $\frac{0}{0}$ type limits at $(0, 0)$ is to rewrite $f(x, y)$ in terms of r and θ (in polar form). Another approach that sometimes works involves the squeeze lemma. We'll discuss these in class.
- Another way to think about the above definition is to say that as (x, y) approaches (a, b) along *any* path, $f(x, y)$ approaches L. (This is really the same as the definition for functions of 1 variable, but in that case you only have two ways to approach a point: from the left and from the right.) More precisely, to compute "the limit along a path", you parametrize a path by $t \mapsto (x(t), y(t))$ with $(x(t_0), y(t_0)) = (a, b)$ and take $\lim_{t\to t_0} f(x(t), y(t))$.
- This reinterpretation of the definition is useful for proving that a limit doesn't exist, by exhibiting two paths along which the limits are different. But it is useless for proving that a limit does exist, since you can't check every path.

Continuity.

- A function $f(x, y)$ is *continuous at* (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b).$$

  We say that f *is continuous* if it is continuous at all points in its domain.

- Sums, products and compositions of continuous functions are continuous (and so are quotients, where the denominator isn't zero) – for example, $|x - y|e^{x+y}$.
- By the definition of contnuity, you can compute the limit of a continuous function by plugging in (a, b).