MATH 233 LECTURE 14 (§14.3): PARTIAL DERIVATIVES

- Given a 2-variable function $f(x,y)$, define

$$\frac{\partial f}{\partial x}(x,y) := \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}\ , \qquad \frac{\partial f}{\partial y}(x,y) := \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}\ .$$

  We also use the notation $f_x(x,y)$, $f_y(x,y)$.

- These represent slopes in the x- and y-directions. More precisely, if you slice the graph of $z = f(x,y)$ by the plane $y = y_0$, you get a curve (the graph of $z = f(x,y_0)$) in this plane. The slope of the tangent line to this curve at $(x_0, y_0, f(x_0,y_0))$ is $\frac{\partial f}{\partial x}(x_0,y_0)$. This tangent line is parametrized by $t \mapsto (x_0+t,\ y_0,\ f(x_0,y_0)+f_x(x_0,y_0)t)$.
- Stupid examples: if $f(x,y) = x^a y^b$, then $\frac{\partial f}{\partial x} = ax^{a-1}y^b$, $\frac{\partial f}{\partial y} = bx^a y^{b-1}$. If $f(x,y) = e^x$, then $\frac{\partial f}{\partial x} = e^x$, $\frac{\partial f}{\partial y} = 0$. When taking $\frac{\partial}{\partial x}$, you have to view y as a constant, which is consistent with the geometric meaning of the partial derivative just described.
- Just as with ordinary derivatives, you may iterate partial derivatives: $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$, $f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$.
- Clairaut's theorem: if both f_{xy} and f_{yx} are continuous, then they are equal – i.e. the order in which you take partial derivatives doesn't matter.

Partial differential equations.

- of fundamental importance in mathematical physics, finance, etc.
- Laplace equation: $f_{xx} + f_{yy} = 0$. Solutions are called *harmonic* functions (e.g. voltage in the absence of a potential field).
- Heat equation: $f_t = \alpha^2 f_{xx}$. (Here f is a function of time t and position x.) Rate of change of (say) temperature is proportional to its concavity at a point.

- Wave equation: $f_{tt} = a^2 f_{xx}$. Satisfied by propagating waves, as the name would imply!
- many other famous examples (Navier-Stokes, Black-Scholes, Schrödinger, not to mention the plethora of such equations in general relativity and quantum field theory...)