# MATH 233 LECTURE 17 (§14.6): DIRECTIONAL DERIVATIVES

## Gradients.

- The gradient of a differentiable function $f(x, y)$ of two variables is a vector-valued function of two variables: that is, you should visualize a vector at each point, rather like one of those meteorological diagrams depicting wind on a map by lots of tiny arrows. The fancy term for this is a *vector field*. Alternatively, you can just think of it as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$.
- The formula for (or rather, definition of!) the gradient is simply

$$(\vec{\nabla} f)(x, y) := \langle f_x(x, y), f_y(x, y)\rangle.$$

- Let $C$ be the level curve of $f$ through $(x_0, y_0)$. Then $(\vec{\nabla} f)(x_0, y_0)$ (the gradient vector at that point) is perpendicular (normal) to $C$.
- You can do the same in three variables: given $F(x, y, z)$, put $(\vec{\nabla} F)(x, y, z) := \langle F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)\rangle$. At any point $(x_0, y_0, z_0)$, $(\vec{\nabla} F)(x_0, y_0, z_0)$ is normal to the level surface of $F$ through that point.

## Directional derivatives.

- By the chain rule, if a bug follows the path $(x(t), y(t))$ and wants to see how fast the function $f(x, y)$ is changing over its head at a point $(x_0, y_0) = (x(t_0), y(t_0))$, it writes

$$\frac{df}{dt}(t_0) = f_x(x_0, y_0)x'(t_0) + f_y(x_0, y_0)y'(t_0).$$

- If it is a very clever bug and moves at unit speed – i.e., if $\langle x'(t_0), y'(t_0)\rangle$ is a unit vector $\hat{u}$ – then this is the slope of the function in its direction of travel. Notice that this is the dot product of the gradient and $\hat{u}$.

- Therefore we define the *directional derivative* of $f$ at $(x_0, y_0)$ in the direction of $\hat{u}$ to be

$$(D_{\hat{u}}f)(x_0, y_0) := (\vec{\nabla} f)(x_0, y_0) \cdot \hat{u}.$$

- What if you want the directional derivative in the direction of an arbitrary vector $\vec{v}$? Is this $(\vec{\nabla} f) \cdot \vec{v}$? Nope! *You must always normalize $\vec{v}$ to a unit vector before trying to compute the directional derivative.* Why? The direction of $\vec{v}$ is the same as the direction of $\hat{u} := \frac{\vec{v}}{\|\vec{v}\|}$, and the directional derivatives are the same. Just because $\vec{v}$ is longer, that doesn't change what the slope is in that direction! So the answer is $(\vec{\nabla} f) \cdot \frac{\vec{v}}{\|\vec{v}\|}$.
- The directional derivative of $f$ at a point $(x_0, y_0)$ is greatest in the direction of the gradient $\vec{\nabla} f$, with magnitude $\left\|\vec{\nabla} f\right\|$. (That is, $\vec{\nabla} f$ is the direction of steepest ascent. The direction of steepest descent is $-\vec{\nabla} f$.) So now you know what the vector field/wind diagram means.