MATH 233 LECTURE 20 (§14.7): OPTIMIZATION

- Recall how to find the *local* maxima and minima of a continuous function $f(x, y)$ of two variables: first, find all critical (stationary and singular) points, usually just stationary points (set $f_x = 0 = f_y$ and solve). Now write $D(x, y) := f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2$. For each stationary point (x_0, y_0), apply the *second derivative test*. That is, evaluate $D(x_0, y_0)$: if it's negative, (x_0, y_0) is a saddle point of f; if it's positive, then (x_0, y_0) is a local maximum (resp. minimum) of f if $f_{xx}(x_0, y_0)$ is negative (resp. positive); if $D(x_0, y_0)$ is zero, the test is inconclusive. (For any singular points, you obviously can't use the second derivative test and simply have to draw the picture.)
- What about *global* maxima and minima of a continuous function $f(x, y)$ on a subset $S \subseteq \mathbb{R}^2$? It turns out that when S is closed (contains all its boundary points) and bounded (contained in some disk), a global maximum and minimum do exist. (For instance: there does exist an (x_0, y_0) – the "global maximum" – such that $f(x, y) \leq f(x_0, y_0)$ for every $(x, y) \in S$.) If S isn't closed or isn't bounded (for instance, if $S = \mathbb{R}^2$), then a global maximum or minimum may or may not exist.
- If S is closed and bounded, the global extrema have to occur (i) on the boundary ∂S of S, or (ii) at critical (stationary and singular) points of f on S. So the game is to first (Step I) find the maximum and minimum value of f on ∂S, then (Step II) evaluate f at all critical points, then (Step III) pick which is the largest value.
- A variant of this type of maximization/minimization problem involves *constraints*: that is, you start with a function (say) of 3 variables, $F(x, y, z)$ that

you want to (say) maximize subject to a constraint $G(x, y, z) = 0$. The idea is to first rewrite the problem as a 2-variable problem of the above type, by solving for $z = z(x, y)$ in the constraint equation and replacing F by $f(x, y) = F(x, y, z(x, y))$.

- Sometimes you will be asked to maximize or minimize on unbounded or sets without boundary (like $x^2 + y^2 < 1$). If a maximum or minimum exists in this case, it has to be at a critical point. By examining the derivatives of the function and sketching the graph (mentally or on paper), you can decide whether the local maxima/minima you find are in fact global.