MATH 233 LECTURE 23: FINISHING UP DIFFERENTIAL MULTIVARIABLE CALCULUS

This lecture ties up two loose ends:

- Clairaut's Theorem: $f_{xy} = f_{yx}$ wherever the second partials are continuous.

Suppose they are continuous near $(0, 0)$.[1] We have

$$f_x(0, y) = \lim_{\Delta x \to 0} \frac{f(\Delta x, y) - f(0, y)}{\Delta x}$$

and so

$$f_{xy}(0, 0) = \lim_{\Delta y \to 0} \frac{f_x(0, \Delta y) - f_x(0, 0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\left(\lim_{\Delta x \to 0} \frac{f(\Delta x, \Delta y) - f(0, \Delta y)}{\Delta x}\right) - \left(\lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0, 0)}{\Delta x}\right)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{f(\Delta x, \Delta y) - f(\Delta x, 0) - f(0, \Delta y) + f(0, 0)}{\Delta x \Delta y}.$$

By the same argument with x and y reversed, $f_{yx}(0, 0)$ is the same as the last expression, but with $\lim_{\Delta y \to 0}$ and $\lim_{\Delta x \to 0}$ reversed.

Why could switching the order limits be a problem? Consider the limits $\lim_{x \to 0} \lim_{y \to 0}$ and $\lim_{y \to 0} \lim_{x \to 0}$ of $\frac{x^2 - y^2}{x^2 + y^2}$: the first is $+1$, the second -1. But of course, this is a badly behaved function.

Write $\Delta(h) := f(h, h) - f(h, 0) - f(0, h) + f(0, 0)$, and set $g(x) := f(x, h) - f(x, 0)$. We have

$$\frac{\Delta(h)}{h} = \frac{g(h) - g(0)}{h}.$$

[1]There is no generality lost here: if you want to do this at (x_0, y_0), we can always replace $f(x, y)$ by $f(x + x_0, y + y_0) =: F(x, y)$.

By the Mean Value Theorem (for g), this equals $g'(a)$ for some $a \in [0, h]$, which equals $f_x(a, h) - f_x(a, 0)$ by definition. Setting $G(y) := f_x(a, y)$, this says that

$$\frac{\Delta(h)}{h} = G(h) - G(0).$$

Using the Mean Value Theorem again (for G),

$$\frac{\Delta(h)}{h^2} = \frac{G(h) - G(0)}{h} = G'(b) = f_{xy}(a, b)$$

for some $b \in [0, h]$. By continuity of f_{xy}, we therefore have

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = \lim_{(a,b) \to (0,0)} f_{xy}(a, b) = f_{xy}(0, 0).$$

An exactly symmetric argument (swapping x and y) yields

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(0, 0).$$

So $f_{xy}(0, 0) = f_{yx}(0, 0)$.

We also had postponed justifying

- Theorem 2: f is differentiable wherever f_x and f_y are continuous.

Suppose f_x and f_y are continuous about $(0, 0)$. We want to show that

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - f(0, 0) - f_x(0, 0)x - f_y(0, 0)y}{\sqrt{x^2 + y^2}} = 0,$$

i.e. that f is differentiable at $(0, 0)$. By the Mean Value Theorem, there exist $a \in [0, x]$ and $b \in [0, y]$ so that

$$f(x, y) - f(0, 0) = \{f(x, y) - f(0, y)\} + \{f(0, y) - f(0, 0)\}$$

$$= f_x(a, y)x + f_y(0, b)y$$

$$= f_x(0, 0)x + \{f_x(a, y) - f_x(0, 0)\}x + f_y(0, 0)y + \{f_y(0, b) - f_y(0, 0)\}y.$$

Plugging this in above yields

$$\lim_{(x,y)\to(0,0)} \left\{ (f_x(a,y) - f_x(0,0)) \frac{x}{\sqrt{x^2+y^2}} + (f_y(0,b) - f_y(0,0)) \frac{y}{\sqrt{x^2+y^2}} \right\}.$$

Why is this zero? Consider

$$\lim_{(x,y)\to(0,0)} (f_x(a,y) - f_x(0,0)) \frac{x}{\sqrt{x^2+y^2}}.$$

By taking (a,y) close enough to $(0,0)$, we can make $f_x(a,y) - f_x(0,0)$ as close to 0 as we like, since $f_x(x,y)$ is continuous near $(0,0)$ by assumption and the limit forces $a \to 0$. Moreover, $\sqrt{x^2+y^2} \geq \sqrt{x^2} = |x|$, and so

$$-1 \leq \frac{x}{\sqrt{x^2+y^2}} \leq 1.$$

Applying the squeeze lemma to the product, and a similar argument to the other term, we get 0.