# MATH 233 LECTURE 25:
# DOUBLE INTEGRALS (HOW TO COMPUTE)

- Let $R = [a, b]\times[c, d]$, $f$ a function on $R$. The volume interpretation of $\iint_R f(x, y)dA$ from the last lecture can be used to compute directly some simple integrals (integral of constant function is volume of a box, etc.). We would like something more useful though. For simplicity assume $f \geq 0$.
- Imagine you are thinly slicing the solid $S$ described by $0 \leq z \leq f(x, y)$ over $R$, parallel to the $xz$-plane, into solid slices $S_j$ of width $\Delta y = \frac{d-c}{n}$. Let $A(y)$ be the area of the slice over $y$, and select $y_j^* \in [y_{j-1}, y_j]$ (notation as in last lecture). Then the volume $V(S) \approx \sum_{j=1}^n V(S_j) \approx \sum_{j=1}^n A(y_j^*)\Delta y$. Taking a limit gives $V(S) = \int_c^d A(y)dy$. But $A(y) = \int_a^b f(x, y)dx$ by single-variable calculus, and so

$$\iint_R f(x, y)dA = V(S) = \int_c^d \left( \int_a^b f(x, y)dx \right) dy.$$

- If we repeat the argument with slices parallel to the $yz$-plane, we get

$$\iint_R f(x, y)dA = \int_a^b \left( \int_c^d f(x, y)dy \right) dx.$$

  The right-hand side of both of these equations is called an *iterated integral.* The equality (that double integrals can be computed by iterated integrals) is called *Fubini's theorem.*
- Average values: another interpretation of the double integral is given by

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)dA,$$

  where $A(R) = (b - a)(d - c)$ is the area of $R$. (You can show that this is the limit of the Riemann sums $\frac{1}{mn} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*)$.)