Let $D$ be a closed, bounded region in the $xy$-plane, with piecewise smooth boundary. Take $R = [a, b] \times [c, d]$ to be a rectangle containing $D$. If $f(x, y)$ is a function on $D$, construct a function on $R$ by

$$F(x, y) := \begin{cases} 
  f(x, y) & \text{if } (x, y) \in D \\
  0 & \text{if } (x, y) \notin D.
\end{cases}$$

The integral of $f$ over $D$ is then defined to be the integral of $F$ over $R$.

How do we evaluate such integrals? It depends on $D$. For instance, we shall say that $D$ is $y$-simple if each line parallel to the $y$-axis intersects $D$ in a single interval (or a point, or not at all) – that is, if there are functions $\phi_1, \phi_2$ on $[a, b]$ such that $D = \{(x, y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$ is the region sandwiched between their graphs. In this case,

$$\iint_D f(x, y) dA = \iint_R F(x, y) dA = \int_a^b \left( \int_c^d F(x, y) dy \right) dx$$

$$= \int_a^b \left( \int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy \right) dx.$$

Similarly, if $D$ is $x$-simple, i.e. $D = \{(x, y) \mid \psi_1(y) \leq x \leq \psi_2(y)\}$, then

$$\iint_D f(x, y) dA = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} f(x, y) dx \right) dy.$$

Some regions are both $x$-simple and $y$-simple, and so you have to judge which is the easier way to perform the iteration, just as in the rectangular case. Some regions are neither $x$- nor $y$-simple, but can be cut up into such regions, and the integrals summed at the end.
• Beware of switching the order of integration in these no-rectangular cases: this requires drawing the region. This will be discussed more in the next class.