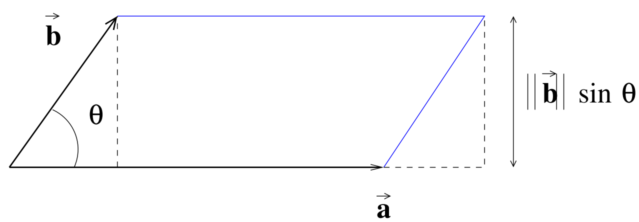
MATH 233 LECTURE 3 (§§12.4-5): MORE ON CROSS-PRODUCTS; LINES AND PLANES IN SPACE

Cross-products, cont'd.

- The area of a parallelogram is given by $\|\vec{a}\| \left(\left\|\vec{b}\right\| \sin\theta \right) = \left\|\vec{a} \times \vec{b}\right\|$, and the area of a triangle is half that:



  Use this formula by finding the vectors along adjacent sides (of a triangle or parallelogram) and taking their cross-product.

- The scalar triple-product (of three vectors in $\mathbb{R}^3$) is given by

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \tag{1}$$

  Its absolute value gives the volume of the parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$. The vectors are coplanar if and only if (1) is zero.

- The cross product is neither associative ($\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$) nor commutative, but is anti-commutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. It satisfies the distributive law, and there are also the "exotic" identities

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot c \,, \qquad \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.$$

  (You don't need to memorize these.)

- In physics, torque (angular force) is computed by $\vec{r} \times \vec{F}$, where $\vec{F}$ is the force vector and $\vec{r}$ the radius vector (pointing from the hinge to where the force is applied). It points *perpendicular* to the plane of motion (i.e. along the axis of motion).

Lines in space.

- parametric equation of the line through $P(x_0, y_0, z_0)$ with direction $\vec{v} = \langle a, b, c \rangle$:

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc.$$

- symmetric equation of the line through P with direction $\vec{v}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

- you should be able to find the parametric and symmetric equations of the line through two given points, or the line through a given point and parallel to a given line.

Planes in space.

- equation of plane $\mathbb{P}$ with normal (perpendicular) vector $\vec{n} = \langle a, b, c \rangle$:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

- use this in conjunction with the cross-product to find the equation of a plane containing 2 vectors (or 3 points).
- what about the plane through a given point and containing a given line? or through a given point and parallel to another given plane?