**MATH 233 LECTURE 30: APPLICATIONS OF DOUBLE INTEGRALS**

- Mass: consider a lamina (flat sheet) covering a region $D$ in the $xy$-plane, with mass density function $\rho : D \to \mathbb{R}$. The total mass $m$ of the lamina is given by $\iint_D \rho(x, y) \, dA$.
- Center of mass: define “moments” of the lamina about the $y$- and $x$-axes by $M_y = \iint_D x \rho(x, y) \, dA$, $M_x = \iint_D y \rho(x, y) \, dA$. The center of mass is then given by $(\bar{x}, \bar{y}) := (M_y/m, M_x/m)$.
- Moment of inertia: the kinetic energy of a particle of mass $m$ traveling on a circle of radius $r$ with angular velocity $\omega$ is given by $\frac{1}{2}mr^2 \omega^2$. The portion $I := mr^2$ is called the moment of inertia. By integrating, we can define moments of inertia of a lamina about the $y$-axis, $x$-axis, and origin: $I_y = \iint_D x^2 \rho(x, y) \, dA$, $I_x = \iint_D y^2 \rho(x, y) \, dA$, $I_0 = \iint_D (x^2 + y^2) \rho(x, y) \, dA$. These measure how hard it is to change the angular velocity of the lamina about the $y$-axis, $x$-axis, and origin.

**Probability.**

- Let $X$ be a random variable with probability density (distribution) function $f : \mathbb{R} \to \mathbb{R}$, then the probability that $X$ lies in the interval $[a, b]$ is given by $P(a \leq X \leq b) = \int_a^b f(x) \, dx$. (Of course, the integral over all of $\mathbb{R}$ must give 1, or 100%).
- Expected value: $\bar{X} := \int_{-\infty}^{\infty} x f(x) \, dx$.
- More generally, if $X$ and $Y$ are random variables with joint probability density function $f : \mathbb{R}^2 \to \mathbb{R}$, then the probability that $X$ and $Y$ lie in some region $D$ is just $P((X, Y) \in D) = \iint_D f(x, y) \, dA$. For example, if this region is $D =$
\{(x, y) \mid x \leq y\}, then this integral computes the probability that \(X\) is smaller than (or equal to) \(Y\).

- Expected values: \(\bar{X} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x, y) \, dx \, dy\), \(\bar{Y} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x, y) \, dx \, dy\).

- Independence: if the joint probability density function is a product, \(f(x, y) = F(x)G(y)\), then the two variables are independent: that is, the probability of \(X\) being in some range is independent of the value of \(Y\).