MATH 233 LECTURE 31: CHANGE OF VARIABLE

- Recall how u-substitutions go in single-variable integrals: given a continuously differentiable function $u \mapsto x(u)$ mapping $[A, B]$ to $[a, b]$ in 1-to-1 fashion, we have
$$\int_a^b f(x)dx = \int_A^B f(x(u))\frac{dx}{du}du.$$
- Next, suppose that $T : D \to D'$ is a 1-to-1, continuously differentiable map of regions, sending
$$(u, v) \mapsto T(u, v) = (x(u, v),\, y(u, v)).$$
Then for any continuous function $f : D \to \mathbb{R}$, we have
$$\iint_D f(x, y)dA = \iint_{D'} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA',$$
where dA is $dx\,dy$, dA' is $du\,dv$, and
$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$
is the *Jacobian* of T. (The bars mean to take absolute value.)