MATH 233 LECTURE 32:
VECTOR FIELDS

• These are vector-valued functions of several real variables. You should visualize a continuum of arrows in the plane (or in space). Mathematically, they are functions from \( \mathbb{R}^2 \to \mathbb{R}^2 \) (or \( \mathbb{R}^3 \to \mathbb{R}^3 \)), and so may be considered as a pair (or triple) of the multivariable functions we have been studying.

Notation: \( \mathbf{F}(x, y) = \langle f(x, y), g(x, y) \rangle = f(x, y)\hat{i} + g(x, y)\hat{j} \) (or \( \mathbf{F}(x, y, z) = \langle f(x, y, z), g(x, y, z), h(x, y, z) \rangle \)).

• An example you have already seen is the gradient of a function. In physics they arise as, for example, velocity fields (think of wind or another fluid) and force fields (gravitational, magnetic, electric).

• The flow lines of a velocity field \( \mathbf{F} \) are the paths followed by a particle whose velocity at any point \((x, y)\) is \( \mathbf{F}(x, y) \). (For example, the flow lines of \( \mathbf{F}(x, y) = -y\hat{i} + x\hat{j} \) are circles.) Parametrizing a flow line by \((x(t), y(t))\) leads to the equation \( \langle x'(t), y'(t) \rangle = \mathbf{F}(x(t), y(t)) \).

• A vector field \( \mathbf{F} \) is called conservative if it is the gradient \( \nabla f \) of a function. Moving all the way around a closed curve (like a circle) against a conservative force field conserves energy, hence the terminology. Conservative fields are rather special: for example, \( \mathbf{F}(x, y) = x\hat{j} \) is not conservative. (Suppose \( \mathbf{F} = \nabla f \): can you find a contradiction?)