• Line integrals are integrals over curves, and are sometimes also called “path
integrals”. More precisely, we will integrate over an oriented curve, which is a
plane (or space) curve together with a choice of direction.

• Let $f : D \to \mathbb{R}$ be a function whose domain includes $C$. Chopping $C$ into
$n$ subarcs of (arc)length $(\Delta s)_i$, and letting $(x^*_i, y^*_i)$ be a sample point on the
$i$th subarc, we define the line integral (with respect to arclength) as a limit of
Riemann sums:

$$\int_C f(x, y) ds := \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i, y^*_i)(\Delta s)_i.$$ 

• To actually calculate the line integral, we will need to choose a smooth parametriza-
tion of $C$. Recall that this is a continuously differentiable function $\vec{r} : [a, b] \to \mathbb{R}^2$
with image $C$, $\vec{r}(a) = \vec{A}$ and $\vec{r}(b) = \vec{B}$ (where $\vec{A}$ and $\vec{B}$ are the endpoints of $C$),
and such that $\vec{r}'(t)$ is never zero on $[a, b]$. Then we have

$$\int_C f(x, y) ds = \int_{a}^{b} f(\vec{r}(t)) \|\vec{r}'(t)\| dt,$$

which (for example) for a plane curve $\vec{r}(t) = (x(t), y(t))$ becomes $\int_{a}^{b} f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}$

• Though we use a choice of parametrization to compute the line integral, its
value is independent of the choice we make. (Notice in particular that if $f$ is
identically 1, then the line integral just gives the arclength of $C$.) In contrast,
if two different curves $C$ and $C'$ have the same endpoints $\vec{A}$ and $\vec{B}$, then the
integrals of $f(x, y)$ over them may well be different.
• Properties: (a) if \( C = C_1 + C_2 \) (two smaller arcs joined at a point), then
\[
\int_C = \int_{C_1} + \int_{C_2};
\]
(b) (writing \(-C\) for \( C \) traced backwards) \( \int_{-C} = -\int_C \); (c)
\[
\int_C (af(x, y) + bg(x, y))\,ds = a\int_C f(x, y)\,ds + b\int_C g(x, y)\,ds.
\]
• There are (in the plane) 2 further kinds of line integrals we will need:
\[
\int_C f(x, y)\,dx := \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)(\Delta x)_i = \int_a^b f(\vec{r}(t))x'(t)\,dt
\]
and
\[
\int_C f(x, y)\,dy := \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)(\Delta y)_i = \int_a^b f(\vec{r}(t))y'(t)\,dt,
\]
which once again don’t depend on the choice of parametrization (and satisfy the
above properties).