MATH 233 LECTURE 33:
LINE INTEGRALS

- Line integrals are integrals over curves, and are sometimes also called "path integrals". More precisely, we will integrate over an oriented curve, which is a plane (or space) curve together with a choice of direction.
- Let $f : D \to \mathbb{R}$ be a function whose domain includes C. Chopping C into n subarcs of (arc)length $(\Delta s)_i$, and letting (x_i^*, y_i^*) be a sample point on the i^{th} subarc, we define the *line integral* (with respect to arclength) as a limit of Riemann sums:

$$\int_C f(x,y)ds := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)(\Delta s)_i.$$

- To actually calculate the line integral, we will need to choose a *smooth parametrization* of C. Recall that this is a continuously differentiable function $\vec{r} : [a,b] \to \mathbb{R}^2$ with image C, $\vec{r}(a) = \vec{A}$ and $\vec{r}(b) = \vec{B}$ (where $\vec{A}$ and $\vec{B}$ are the endpoints of C), and such that $\vec{r}'(t)$ is never zero on $[a,b]$. Then we have

$$\int_C f(x,y)ds = \int_a^b f(\vec{r}(t)) \left\| \vec{r}'(t) \right\| dt,$$

which (for example) for a plane curve $\vec{r}(t) = \langle x(t), y(t)\rangle$ becomes $\int_a^b f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t)$

- Though we use a choice of parametrization to compute the line integral, its value is independent of the choice we make. (Notice in particular that if f is identically 1, then the line integral just gives the arclength of C.) In contrast, if two different curves C and C' have the same endpoints $\vec{A}$ and $\vec{B}$, then the integrals of $f(x,y)$ over them may well be different.

- Properties: (a) if $C = C_1 + C_2$ (two smaller arcs joined at a point), then $\int_C = \int_{C_1} + \int_{C_2}$; (b) (writing $-C$ for C traced backwards) $\int_{-C} = -\int_C$; (c) $\int_C (af(x,y) + bg(x,y))ds = a\int_C f(x,y)ds + b\int_C g(x,y)ds$.
- There are (in the plane) 2 further kinds of line integrals we will need:

$$\int_C f(x,y)dx := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)(\Delta x)_i = \int_a^b f(\vec{r}(t))x'(t)dt$$

  and

$$\int_C f(x,y)dy := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)(\Delta y)_i = \int_a^b f(\vec{r}(t))y'(t)dt,$$

  which once again don't depend on the choice of parametrization (and satisfy the above properties).