MATH 233 LECTURE 34: LINE INTEGRALS OF VECTOR FIELDS

- Begin with a particle moving in the plane along a curve C (from endpoint A to endpoint B), under the influence of a force field $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$. To compute the work done by $\vec{F}$, we chop C into pieces of (arc)length $(\Delta s)_i$ with sample points (x_i^*, y_i^*), and write $\hat{T}(x,y)$ for the unit tangent vector to C at a point (x,y) on C; the bit of work along this arc is approximated by $(\Delta W)_i \approx \vec{F}(x_i^*, y_i^*) \cdot \hat{T}(x_i^*, y_i^*)(\Delta s)_i$. (Here "$\cdot$" is dot product.) Taking the limit of the (Riemann) sum of these $(\Delta W)_i$ gives

$$W = \int_C \vec{F}(x,y) \cdot \hat{T}(x,y)\, ds,$$

a line integral. Notice that we have made no reference to a parametrization in defining this.

- A shorthand for the right-hand side of this last formula is

$$\int_C \vec{F} \cdot \hat{T}\, ds.$$

It is called the *line integral of $\vec{F}$ along C*, and the above result says it computes the work done by the force field $\vec{F}$ on the particle as it moves along C from endpoint to endpoint. Of course, we don't always want to think of the vector field $\vec{F}$ as a force field, or of the line integral as computing work; the integral makes sense on its own without these interpretations.

- To actually compute the integral, we need to choose a parametrization $\vec{r}(t) = \langle x(t), y(t) \rangle$, $t \in [a,b]$, of C. Using this, we get

$$W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}\, \|\vec{r}\,'(t)\|\, dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) dt$$

which we will sometimes write in the shorthand

$$\int_a^b \vec{F} \cdot d\vec{r} \;\text{ or }\; \int_C \vec{F} \cdot d\vec{r}.$$

Or we can expand it by writing $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$, which gives

$$\int_a^b \left(P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right) dt$$

$$= \int_a^b P\,dx + Q\,dy.$$

- Suppose next that $\vec{F} = \vec{\nabla} f$; that is, that $\vec{F}$ is conservative. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_a^b \left((\vec{\nabla} f)(\vec{r}(t))\right) \cdot \vec{r}'(t) dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right) dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt = \left[f(\vec{r}(t))\right]_a^b$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(B) - f(A),$$

where A and B are the endpoints of C. This is the *Fundamental Theorem of Calculus for Line Integrals.* In particular, it says that the value of the line integral of a *conservative* vector field depends only upon the endpoints of the curve C of integration — that is, it is independent of the choice of path from A to B.