MATH 233 LECTURE 35: CONSERVATIVE VECTOR FIELDS

- Consider a vector field $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ on a region (connected open set) D in $\mathbb{R}^2$. If $\vec{F} = \vec{\nabla} f$ — that is, if $\vec{F}$ is conservative — then $\int_C \vec{F} \cdot d\vec{r}$ only depends on the endpoints A and B of C (independence of path). In particular, if C is closed ($A = B$), then $\int_C \vec{F} \cdot d\vec{r} = 0$.
- Moreover, if $\vec{F}$ is conservative, then $P = f_x$, $Q = f_y$, and so by Clairaut's theorem, $P_y = f_{xy} = f_{yx} = Q_x$.
- To state a converse to this last result, suppose D is simply connected: this means that it has no holes. Then $P_y = Q_x$ implies that $\vec{F}$ is conservative. We will check this if D is a rectangle.
- If D has a hole, then it is possible to have $P_y = Q_x$ and $\vec{F}$ still fail to be conservative.
- For instance, $\vec{F} = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$ satisfies $P_y = \frac{x^2-y^2}{(x^2+y^2)^2} = Q_x$, but if C is the (closed) circle of radius 1, then $\int_C \vec{F} \cdot d\vec{r} = 2\pi \neq 0$, so $\vec{F}$ can't be conservative. The problem is that the domain of definition of $\vec{F}$ omits the origin, hence has a hole. We are saying there can't be a function f on $D = \mathbb{R}^2 - \{0\}$ such that $\vec{F} = \vec{\nabla} f$ on all of D. If we take a smaller region D' inside D which doesn't have a hole, like a disk of radius 1 about $(2,0)$, then the restriction of $\vec{F}$ to D' is indeed conservative (and obviously D' doesn't contain C, so there is no contradiction).
- For a vector field $\vec{F}$ on any region D, $\vec{F}$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r}$ is independent of path (or equivalently, zero on all closed loops).
- We will discuss in lecture how to actually find f in the event that $\vec{F}$ is conservative.